

# Dual-gauge and co-BRST symmetries in Abelian gauge theories

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**Abstract:** Taking the simple examples of an Abelian 1-form gauge theory in two (1+1)-dimensions, a 2-form gauge theory in four (3+1)-dimensions and a 3-form gauge theory in six (5+1)-dimensions of spacetime, we establish that such gauge theories respect, in addition to the gauge symmetry transformations that are generated by the first-class constraints of the theory, additional continuous symmetry transformations. We christen the latter symmetry transformations as the dual-gauge symmetry transformations. We generalize the above gauge- and dual-gauge symmetry transformations to obtain the proper (anti-)BRST and (anti-)dual-BRST transformations for the Abelian 3-form gauge theory within the framework of BRST formalism. We claim that any arbitrary Abelian  $p$ -form ( $p = 1, 2, 3, \dots$ ) gauge theory would respect the above cited additional symmetry in  $D = 2p$  dimensions of spacetime. By exploiting the above inputs, we establish that the Abelian 3-form gauge theory, in six (5+1)-dimensions of spacetime, is a model for the Hodge theory whose discrete and continuous symmetry transformations provide the physical realizations of the de Rham cohomological operators of differential geometry. As far as the physical utility of the above nilpotent symmetries is concerned, we demonstrate that our present theory is a field theoretic model for the quasi topological field theory (q-TFT).

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# 1 Introduction

It has been well-established that the four  $(3 + 1)$ -dimensional (4D) non-Abelian 1-form ( $A^{(1)} = dx^\mu A_\mu$ ) gauge theories are at the heart of standard model of high energy physics where there is a stunning degree of agreement between theory and experiment. Two of the central shortcomings of the standard model of particle physics are the experimental observation of the mass of the neutrinos and, so far, no concrete experimental detection of the Higgs particles which provide masses to the gauge bosons and fermions of the standard model of particle physics. It has been possible to construct models that provide masses to the (non-)Abelian 1-form gauge bosons without taking any recourse to the Higgs mechanism. These models are based on the merging of 1-form and 2-form [ $B^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu)B_{\mu\nu}$ ] gauge fields through the topological coupling [1-4]. In an exactly similar fashion, the 2-form gauge boson has been found to acquire a mass through the topological coupling with the 3-form gauge field (see, e.g., [5]). Thus, there has been a renewed interest in the study of the higher  $p$ -form ( $p = 2, 3, 4, \dots$ ) gauge theories. One of the central purposes of our present investigation is to discuss some *novel* continuous and discrete symmetry transformations that are associated with the 6D Abelian 3-form gauge theory.

In recent years, it has become fashionable to study higher  $D$ -dimensional [ $(D - 1) + 1$ ] (with  $D = 5, 6, 7, \dots$ )  $p$ -form tensor gauge fields because these fields appear in the quantum excitations of the (super)string theories and related extended objects (see, e.g., [6-8]). In fact, as is well-known, the quantum versions of the (super)strings themselves live in dimensions of spacetime that are higher than the physical four  $(3 + 1)$ -dimensions of spacetime. Thus, from the point of view of the modern developments in (super)string theories, it is important to study higher  $p$ -form gauge theories in higher dimensions ( $D > 4$ ) of spacetime. There is yet another motivation to study, particularly, higher Abelian  $p$ -form ( $p \geq 2$ ) gauge theories in higher dimensions ( $D > 4$ ) of spacetime. In our very recent paper on the existence of the (anti-)dual Becchi-Rouet-Stora-Tyutin (BRST) symmetries [9], we have claimed that the dual-gauge- and (anti-)dual BRST symmetries would always exist for any arbitrary Abelian  $p$ -form gauge theory in the specific  $D$ -dimensions of spacetime (when  $D = 2p$ ). In other words, we have proposed that, besides the usual gauge- and corresponding nilpotent (anti-)BRST symmetries, any arbitrary Abelian  $p$ -form gauge theory would be *also* endowed with the dual-gauge- and corresponding nilpotent (anti-)dual BRST symmetry transformations in the spacetime dimensions  $D = 2p$ .

To follow the current trends and to corroborate the above assertions, we have chosen the 6D Abelian 3-form gauge theory to demonstrate that it respects the dual-gauge- and nilpotent (anti-)dual BRST [or (anti-)co-BRST] symmetries along with the *usual* gauge- and corresponding nilpotent (anti-)BRST symmetries. As a consequence, the present theory provides a tractable field theoretic model for the Hodge theory in six  $(5 + 1)$ -dimensions of spacetime as do the Abelian 1-form gauge theory in two  $(1 + 1)$ -dimensions [10-12] and Abelian 2-form gauge theory in four  $(3 + 1)$ -dimensions of spacetime [13-16]. In all the above theories, we have obtained the physical realizations of the de Rham cohomological operators of differential geometry [17-20] in the language of discrete and continuous symmetry transformations. Furthermore, we have deduced the analogue of the celebrated Hodge decomposition theorem in the quantum Hilbert space of states for the above theories.

In our present investigation, we have taken the (anti-)BRST invariant coupled (but

equivalent) Lagrangian densities from our earlier works [21,22] on the superfield approach to BRST formalism for the Abelian 3-form gauge theory in any arbitrary  $D$ -dimensions of spacetime where we have established the connection of the Curci-Ferrari (CF) type restrictions with the geometrical objects called gerbes (see, e.g., [22] for details). It is the specific property of the six (5 +1) dimensional (6D) spacetime that the kinetic term ( $\frac{1}{24}H^{\mu\nu\eta\kappa}H_{\mu\nu\eta\kappa}$ ) of the above Lagrangian densities can be linearized [cf.(30),(31)] by exploiting the 6D Levi-Civita tensor ( $\varepsilon_{\mu\nu\eta\kappa\rho\sigma}$ ). This linearization enables us to derive the dual-gauge- and nilpotent (anti-)dual BRST transformations besides the usual gauge- and off-shell nilpotent (anti-)BRST transformations. We have deduced a bosonic symmetry in the theory which is obtained from the *suitable* anticommutators between (anti-)BRST and (anti-)co-BRST symmetry transformations. We show, in our present endeavor, that there are, in totality, six *useful* continuous symmetries in the theory that include the usual ghost-scale symmetry transformations as well. There also exists a set of discrete symmetry transformations in the theory which plays a very crucial role in our present discussions on the proof of our 6D free Abelian gauge theory to be a tractable model for the Hodge theory.

The physical relevance of all these studies, it may be pointed out, is the observation that the 2D free Abelian 1-form gauge theory provides a new model [23] for the topological field theory (TFT) which captures a part of the key features of Witten type TFT and some salient features of Schwarz type TFT. In an exactly similar fashion, we have been able to prove that the 4D Abelian 2-form gauge theory is a model for the quasi-topological field theory (q-TFT) [24]. We demonstrate, in our present endeavor, that present 6D Abelian 3-form gauge theory is *also* a model of q-TFT apart from being a cute model for the Hodge theory. To study the physical constraints on the theory, we exploit the Hodge decomposition theorem in the quantum Hilbert space of states and choose the physical state to be the *harmonic* state of the theory [which is (anti-)BRST as well as (anti-)co-BRST invariant and, hence, is the most symmetric state in the theory]. We have already chosen such a physical state (i.e. the harmonic state) in our earlier works on the proof of the *exact* topological nature of the free 2D Abelian 1-form gauge theory [25,26].

The following factors have spurred our curiosity and interest in pursuing our present investigation. First, to put our claim [9] on a firmer footing, it is essential for us to show that the 6D Abelian 3-form gauge theory is *also* endowed with the dual-gauge- and off-shell nilpotent (anti-)co-BRST symmetry transformations as we have been able to show in the cases of 2D Abelian 1-form and 4D Abelian 2-form gauge theories. Second, our present exercise helps us to establish that the 6D Abelian 3-form gauge theory is a model for the q-TFT which is similar in contents and textures as the 4D Abelian 2-form gauge theory (see, e.g., [24]). Finally, it is always challenging to explore some new features that turn up in the study of the higher  $D$ -dimensional ( $D > 4$ ) and higher  $p$ -form ( $p \geq 2$ ) gauge theories that are, in some sense, generalizations of the usual 4D gauge theories based on the 1-form potentials (that provide the basis for the standard model of particle physics).

The material of our present investigation is organized as follows. In Sec. II, we discuss the continuous (dual-)gauge transformations and discrete symmetry transformations for the Abelian 1-form, 2-form and 3-form gauge theories in 2D, 4D, 6D spacetime, respectively. We *also* make brief comments on the (anti-)BRST and (anti-)co-BRST symmetry transformations associated with the 2D Abelian 1-form and 4D Abelian 2-form gauge theories in this section. Our Sec. III is devoted to the discussion of (anti-)BRST symmetries and corre-

sponding conserved charges for the 6D Abelian 3-form gauge theory. In Sec. IV, we discuss about the (anti-)co-BRST symmetries and corresponding charges for the above 6D Abelian 3-form gauge theory. We derive the anticommutators of (anti-)BRST and (anti-)co-BRST symmetries in Sec. V and deduce a single bosonic symmetry and its corresponding charge. We take up the ghost-scale symmetry in Sec. VI and derive its conserved charge. In Sec. VII, we discuss the algebraic structures of *all* the conserved charges and devote time on the analysis of cohomological aspects of the above algebraic structures. Our Sec. VIII is devoted to the proof that our present theory is *also* a model for the q-TFT. Finally, we make some concluding remarks and point out a few future directions in Sec. IX.

In our Appendices A, B and C, we discuss some explicit computations that have been used in the main body of our text (i.e. the derivation of coupled Lagrangian densities (30), (31) and the extended BRST algebra [cf. (79)]). Further, in our Appendix D, we discuss very briefly the self-duality condition for the general  $D = 2p$  dimensional Abelian  $p$ -form gauge theory. Our Appendix E is devoted to the discussion of *extra* bosonic symmetries.

*Notations and conventions:* We adopt here the conventions and notations such that the background flat  $D$ -dimensional Minkowskian spacetime manifold is endowed with a metric that has signatures  $(+1, -1, -1, \dots)$  so that the dot product between two non-null vectors  $P_\mu$  and  $Q_\mu$  is:  $P \cdot Q = P_\mu Q^\mu = P_0 Q_0 - P_i Q_i$ , where the Greek indices  $\mu, \nu, \rho, \dots = 0, 1, 2, \dots, D-1$  correspond to the spacetime directions and the Latin indices  $i, j, k, \dots = 1, 2, 3, \dots, D-1$  stand for the space directions only. We also follow the convention:  $(\delta B_{\mu\nu}/\delta B_{\rho\sigma}) = (1/2!)(\delta_\mu^\rho \delta_\nu^\sigma - \delta_\nu^\rho \delta_\mu^\sigma)$ ,  $(\delta A_{\mu\nu\eta}/\delta A_{\rho\sigma\kappa}) = (1/3!)[\delta_\mu^\rho(\delta_\nu^\sigma \delta_\eta^\kappa - \delta_\nu^\kappa \delta_\eta^\sigma) + \delta_\mu^\sigma(\delta_\nu^\kappa \delta_\eta^\rho - \delta_\nu^\rho \delta_\eta^\kappa) + \delta_\mu^\kappa(\delta_\nu^\rho \delta_\eta^\sigma - \delta_\nu^\sigma \delta_\eta^\rho)]$ , etc., where  $B_{\mu\nu}$  and  $A_{\mu\nu\eta}$  are the totally antisymmetric tensor gauge fields.

## 2 Preliminaries: dual-gauge symmetries

In our subsection 2.1, we briefly mention the key points connected with the (dual-)gauge transformations for the Abelian 1-form and 2-form gauge theories in 2D and 4D flat Minkowskian spacetime, respectively [10-16]. In subsection 2.2, we discuss about the continuous (dual-)gauge- and discrete symmetries for the 6D Abelian 3-form gauge theory.

### 2.1 Abelian 1-form and 2-form gauge theories: symmetries

We begin with the two  $(1+1)$ -dimensional (2D) gauge-fixed Lagrangian density for a free Abelian 1-form gauge theory in the Feynman gauge (see, e.g., [10-12])

$$\mathcal{L}_{(1)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial \cdot A)^2 \equiv \frac{1}{2} E^2 - \frac{1}{2} (\partial \cdot A)^2, \quad (1)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is derived from the 2-form  $[F^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu) F_{\mu\nu}]$  which owes its origin to the application of an exterior derivative  $d = dx^\mu \partial_\mu$  (with  $d^2 = 0$ ) on a 1-form ( $A^{(1)} = dx^\mu A_\mu$ ) corresponding to the gauge potential  $A_\mu$ . The gauge-fixing term  $(\partial \cdot A)$  is obtained from the application of a co-exterior derivative  $\delta = - * d *$  (with  $\delta^2 = 0$ ) on the 1-form gauge connection  $A^{(1)}$ . Here  $(*)$  is the Hodge duality operation on the 2D spacetime flat Minkowskian manifold. In 2D spacetime, the only existing component of the curvature tensor  $F_{\mu\nu}$  ( $\mu, \nu, \dots = 0, 1$ ) is the electric field (i.e.  $F_{01} = -F_{10} = E$ ).

The gauge- and the dual-gauge symmetry transformations ( $\delta_g, \delta_{dg}$ ) for the above gauge-fixed Lagrangian density (1) are given by (see, e.g., [10-12] for details)

$$\delta_g A_\mu(x) = \partial_\mu \Omega(x), \quad \delta_{dg} A_\mu(x) = -\varepsilon_{\mu\nu} \partial^\nu \Sigma(x), \quad (2)$$

where  $\Omega(x)$  and  $\Sigma(x)$  are the infinitesimal local gauge- and dual-gauge parameters, respectively, and  $\varepsilon_{\mu\nu}$  is the 2D Levi-Civita tensor with  $\varepsilon_{01} = +1 = -\varepsilon^{01}$ . The latter satisfies  $\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2!$ ,  $\varepsilon_{\mu\nu} \varepsilon^{\mu\lambda} = -\delta_\nu^\lambda$ , etc. It should be noted that, under the infinitesimal gauge ( $\delta_g$ )- and dual-gauge ( $\delta_{dg}$ ) symmetry transformations, the curvature tensor  $F_{\mu\nu}$  and the gauge-fixing term  $(\partial \cdot A)$  remain invariant, respectively. One can check that, under the above infinitesimal transformations (2), the Lagrangian density transforms as

$$\delta_g \mathcal{L}_{(1)} = -(\partial \cdot A) (\square \Omega), \quad \delta_{dg} \mathcal{L}_{(1)} = (E) (\square \Sigma). \quad (3)$$

We note that the Lagrangian density remains invariant under the above symmetry transformations (2) if we impose the conditions  $\square \Omega(x) = 0$  and  $\square \Sigma(x) = 0$  from outside. However, we obtain a *perfect* symmetry invariance of the modified version of the above Lagrangian density within the framework of BRST formalism where the (dual-)gauge symmetry transformations are generalized to their counterparts. The latter symmetries are nothing but the supersymmetric type (on-shell as well as off-shell nilpotent) (co-)BRST and (anti-)BRST symmetry transformations (see, e.g., [10-12]).

In two dimensions of spacetime, the self-duality condition on the Abelian 1-form gauge connection is defined in terms of the Hodge duality (\*) operation as

$$* A^{(1)} = dx^\mu (-\varepsilon_{\mu\nu} A^\nu) \equiv dx^\mu \tilde{A}_\mu. \quad (4)$$

Thus, we observe that  $\tilde{A}_\mu = -\varepsilon_{\mu\nu} A^\nu$ . Under the transformation  $A_\mu \rightarrow \tilde{A}_\mu = -\varepsilon_{\mu\nu} A^\nu$ , it can be checked that  $\mathcal{L}_{(1)} \rightarrow -\mathcal{L}_{(1)}$  because of the fact that  $(\partial \cdot A) \longleftrightarrow E$ . As a consequence, the Lagrangian density  $\mathcal{L}_{(1)}$  is *not* self-duality invariant. However, it is obvious that, under the discrete symmetry transformations  $A_\mu \rightarrow \mp i \varepsilon_{\mu\nu} A^\nu$ , we have  $\mathcal{L}_{(1)} \rightarrow \mathcal{L}_{(1)}$ . These discrete symmetry transformations are at the heart of the existence of dual-gauge symmetry transformations in the theory. The latter provides a physical realization of the Hodge duality (\*) operation of differential geometry [10-12]. Furthermore, these symmetries are the reasons behind the existence of similar kind of restrictions on the (dual-) gauge parameters  $\Sigma(x)$  and  $\Omega(x)$  for the (dual-)gauge invariance of  $\mathcal{L}_{(1)}$  [cf. (3)]. The above continuous and discrete symmetries have been exploited in the case of (non-)interacting 2D Abelian theory within the framework of BRST formalism and these theories have been shown to be the models for Hodge theory (see, e.g., [10-12]) as, the interplay of these symmetries, provide the physical realizations of all aspects of the de Rham cohomological operators of differential geometry [17-20] in terms of the above symmetry transformations.

With the above background, let us look at the 4D free Abelian 2-form gauge theory which is described by the gauge-fixed Lagrangian density (in the Feynman gauge) as

$$\mathcal{L}_{(2)} = \frac{1}{12} H_{\mu\nu\eta} H^{\mu\nu\eta} + \frac{1}{2} (\partial_\mu B^{\mu\nu})(\partial^\eta B_{\eta\nu}), \quad (5)$$

where  $H_{\mu\nu\eta} = \partial_\mu B_{\nu\eta} + \partial_\nu B_{\eta\mu} + \partial_\eta B_{\mu\nu}$  is the totally antisymmetric curvature tensor derived from the 3-form  $[H^{(3)} = dB^{(2)} = (1/3!)(dx^\mu \wedge dx^\nu \wedge dx^\eta) H_{\mu\nu\eta}]$ . The latter is obtained from

the application of the exterior derivative  $d$  on the 2-form  $[B^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu) B_{\mu\nu}]$  antisymmetric ( $B_{\mu\nu} = -B_{\nu\mu}$ ) tensor gauge field  $B_{\mu\nu}$ . The gauge-fixing term (in the Feynman gauge) can be obtained by the action of the co-exterior derivative  $\delta$  on the 2-form gauge field  $[\delta B^{(2)} = (\partial^\nu B_{\nu\mu})dx^\mu]$ . The above Lagrangian density (5) transforms as

$$\delta_g \mathcal{L}_{(2)} = (\partial_\mu B^{\mu\nu}) [\square \Omega_\nu - \partial_\nu(\partial \cdot \Omega)], \quad \delta_d \mathcal{L}_{(2)} = \frac{1}{3!} \varepsilon_{\mu\nu\eta\kappa} H^{\mu\nu\eta} [\square \Sigma^\kappa - \partial^\kappa(\partial \cdot \Sigma)], \quad (6)$$

under the following gauge- and dual-gauge transformations:

$$\begin{aligned} \delta_g B_{\mu\nu} &= (\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu), & \delta_g H_{\mu\nu\eta} &= 0, \\ \delta_{dg} B_{\mu\nu} &= \varepsilon_{\mu\nu\eta\kappa} \partial^\eta \Sigma^\kappa, & \delta_{dg}(\partial^\mu B_{\mu\nu}) &= 0, \end{aligned} \quad (7)$$

where  $\Omega_\mu(x)$  and  $\Sigma_\mu(x)$  are the infinitesimal local gauge- and dual-gauge parameters and  $\varepsilon_{\mu\nu\eta\kappa}$  is the 4D Levi-Civita tensor with  $\varepsilon_{0123} = +1 \equiv -\varepsilon^{0123}$ . Furthermore, we note that  $\varepsilon_{\mu\nu\eta\kappa} \varepsilon^{\mu\nu\eta\kappa} = -4!$ ,  $\varepsilon_{\mu\nu\eta\kappa} \varepsilon^{\mu\nu\eta\lambda} = -3! \delta_\kappa^\lambda$ ,  $\varepsilon_{\mu\nu\eta\kappa} \varepsilon^{\mu\nu\lambda\rho} = -2! (\delta_\eta^\lambda \delta_\kappa^\rho - \delta_\kappa^\lambda \delta_\eta^\rho)$ , etc. The key features of the above continuous (dual-)gauge transformations are the invariance of the gauge-fixing and kinetic terms, respectively. We note that, for the (dual-)gauge invariance in the theory, we have to impose  $\square \Sigma_\mu - \partial_\mu(\partial \cdot \Sigma) = 0$ ,  $\square \Omega_\mu - \partial_\mu(\partial \cdot \Omega) = 0$  in equation (6) from outside. However, these restrictions are *not* required within the framework of BRST formalism and there exists a perfect symmetry in the theory (see, e.g., [13-16]).

The self-duality condition on the 2-form  $B^{(2)}$  can be defined in terms of the Hodge duality  $(*)$  operation (on the 4D flat spacetime manifold) as:

$$* B^{(2)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) \varepsilon_{\mu\nu\eta\kappa} B^{\eta\kappa} \equiv (dx^\mu \wedge dx^\nu) \tilde{B}_{\mu\nu}, \quad \tilde{B}_{\mu\nu} = \frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa} B^{\eta\kappa}. \quad (8)$$

It turns out that the gauge-fixed Lagrangian density (5) respects the discrete symmetry transformations  $B_{\mu\nu} \rightarrow \pm \frac{i}{2} \varepsilon_{\mu\nu\eta\kappa} B^{\eta\kappa}$  (i.e.  $\mathcal{L}_{(2)} \rightarrow \mathcal{L}_{(2)}$ ) because the kinetic and gauge-fixing terms exchange with each-other [i.e.  $\frac{1}{12} H^{\mu\nu\eta} H_{\mu\nu\eta} \leftrightarrow \frac{1}{2} (\partial_\mu B^{\mu\nu})(\partial^\nu B_{\nu\mu})$ ]. It is clear from (8) that the existence of this discrete symmetry transformations owes its origin to the self-duality (Hodge duality) condition. We further note that the above (dual-)gauge continuous symmetry transformations as well as discrete symmetry transformations have been exploited within the framework of BRST formalism and, in our earlier works [13-16], it has been established that the Abelian 2-form gauge theory is a field theoretic model for the Hodge theory in the 4D Minkowskian flat spacetime (as it provides the physical realizations of the de Rham cohomological operators of differential geometry).

## 2.2 Abelian 3-form gauge theories: symmetries

Let us begin with the following gauge-fixed Lagrangian density for the 6D Abelian 3-form gauge theory in the Feynman gauge (see, e.g., [21,22] for details)

$$\mathcal{L}_{(3)} = \frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa} + \frac{1}{2} (\partial_\mu A^{\mu\nu\eta}) (\partial^\rho A_{\rho\nu\eta}), \quad (9)$$

where the curvature tensor  $H_{\mu\nu\eta\kappa} = \partial_\mu A_{\nu\eta\kappa} - \partial_\nu A_{\eta\kappa\mu} + \partial_\eta A_{\kappa\mu\nu} - \partial_\kappa A_{\mu\nu\eta}$  (of the kinetic term for the gauge field  $A_{\mu\nu\eta}$ ) is derived from the following 4-form

$$H^{(4)} = dA^{(3)} = \frac{1}{4!} (dx^\mu \wedge dx^\nu \wedge dx^\eta \wedge dx^\kappa) H_{\mu\nu\eta\kappa}. \quad (10)$$

Here  $d = dx^\mu \partial_\mu$  is the exterior derivative and 3-form  $A^{(3)} = \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\eta) A_{\mu\nu\eta}$  defines the totally antisymmetric tensor gauge field  $A_{\mu\nu\eta}$ . The gauge-fixing term of (9) has its origin in the co-exterior derivative  $\delta = - * d *$  as  $\delta A^{(3)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) (\partial^\eta A_{\eta\mu\nu})$  where  $*$  is the Hodge duality operation on the 6D Minkowskian flat spacetime manifold. We define the following local, infinitesimal (dual-)gauge transformations  $(\delta_{dg}, \delta_g)$ :

$$\begin{aligned} \delta_{dg} A_{\mu\nu\eta} &= \frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \Sigma^{\rho\sigma}, & \delta_{dg}(\partial^\eta A_{\eta\mu\nu}) &= 0, \\ \delta_g A_{\mu\nu\eta} &= \partial_\mu \Omega_{\nu\eta} + \partial_\nu \Omega_{\eta\mu} + \partial_\eta \Omega_{\mu\nu}, & \delta_g(H_{\mu\nu\eta\kappa}) &= 0, \end{aligned} \quad (11)$$

where  $\Sigma_{\mu\nu}(x)$  and  $\Omega_{\mu\nu}(x)$  are the (dual-)gauge parameters in the theory. The totally antisymmetric 6D Levi-Civita tensor satisfies  $\varepsilon_{\mu\nu\eta\kappa\lambda\rho} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} = -6!$ ,  $\varepsilon_{\mu\nu\eta\kappa\lambda\rho} \varepsilon^{\mu\nu\eta\kappa\lambda\sigma} = -5! \delta_\rho^\sigma$ , etc., and we have chosen  $\varepsilon_{012345} = +1 = -\varepsilon^{012345}$ . It is to be noted that the gauge-fixing and kinetic terms, owing their origin to the (co-)exterior derivatives, remain invariant under the (dual-)gauge transformations, respectively. Furthermore, we obtain the following transformations for the Lagrangian density  $\mathcal{L}_{(3)}$  under (11), namely;

$$\begin{aligned} \delta_{dg} \mathcal{L}_{(3)} &= -\frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\mu\nu\eta\kappa} [\square \Sigma^{\rho\sigma} + \partial^\rho (\partial_\lambda \Sigma^{\sigma\lambda}) - \partial^\sigma (\partial_\lambda \Sigma^{\rho\lambda})], \\ \delta_g \mathcal{L}_{(3)} &= (\partial_\mu A^{\mu\nu\eta}) [\square \Omega_{\nu\eta} + \partial_\nu (\partial^\rho \Omega_{\eta\rho}) - \partial_\eta (\partial^\rho \Omega_{\nu\rho})]. \end{aligned} \quad (12)$$

Thus, we observe that, for the (dual-)gauge invariance, exactly similar kind of restrictions must to be imposed on the (dual-)gauge parameters  $\Sigma_{\mu\nu}(x)$  and  $\Omega_{\mu\nu}(x)$ , namely;

$$\begin{aligned} \square \Sigma_{\mu\nu} + \partial_\mu (\partial^\lambda \Sigma_{\nu\lambda}) - \partial_\nu (\partial^\lambda \Sigma_{\mu\lambda}) &= 0, \\ \square \Omega_{\mu\nu} + \partial_\mu (\partial^\lambda \Omega_{\nu\lambda}) - \partial_\nu (\partial^\lambda \Omega_{\mu\lambda}) &= 0. \end{aligned} \quad (13)$$

The reason behind *these* restrictions is the existence of a discrete symmetry invariance in the theory which we elaborate below in an explicit manner.

Let us consider the self-duality condition for the Abelian 3-form connection  $A^{(3)}$  in the language of the Hodge duality ( $*$ ) operation (defined on a 6D Minkowskian manifold):

$$* (A^{(3)}) = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\eta) \tilde{A}_{\mu\nu\eta}, \quad \tilde{A}_{\mu\nu\eta} = -\frac{1}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}. \quad (14)$$

As we have seen the importance of self-duality transformations in the context of Abelian 1-form and 2-form gauge theories, under the following discrete symmetry transformations:

$$A_{\mu\nu\eta} \longrightarrow \pm \frac{i}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}, \quad (15)$$

the Lagrangian density  $\mathcal{L}_{(3)}$  remains invariant. We note, once again, that the symmetry transformations (15) owe their mathematical origin to the self-duality condition (14). In fact, the self-duality condition (14) is the root-cause for the existence of dual-gauge symmetry transformations in the theory *and* the derivation of similar kind of restrictions in (13). We would like to lay emphasis on the fact that the discrete symmetry transformations (15) would provide the physical realizations of the Hodge duality ( $*$ ) operation of differential geometry as we shall see later in the context of BRST formalism (see, Sec. VII below).

We can linearize the kinetic and gauge-fixing terms by invoking the Nakanishi-Lautrup type auxiliary fields  $K_{\mu\nu}$  and  $\mathcal{K}_{\mu\nu}$  as given below:

$$\mathcal{L}_{(K)} = \frac{1}{2} \mathcal{K}^{\mu\nu} \mathcal{K}_{\mu\nu} - \mathcal{K}^{\mu\nu} \left( \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} \right) + K^{\mu\nu} (\partial^\eta A_{\eta\mu\nu}) - \frac{1}{2} K^{\mu\nu} K_{\mu\nu}. \quad (16)$$

The gauge- and dual-gauge transformations  $(\delta_g, \delta_{dg})$ , for the fields of this linearized version of the gauge-fixed Lagrangian density, are

$$\begin{aligned} \delta_g A_{\mu\nu\eta} &= \partial_\mu \Omega_{\nu\eta} + \partial_\nu \Omega_{\eta\mu} + \partial_\eta \Omega_{\mu\nu}, & \delta_g (H_{\mu\nu\eta\kappa}) &= 0, & \delta_g K_{\mu\nu} &= 0, & \delta_g \mathcal{K}_{\mu\nu} &= 0, \\ \delta_{dg} A_{\mu\nu\eta} &= \frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \Sigma^{\rho\sigma}, & \delta_{dg} (\partial^\eta A_{\eta\mu\nu}) &= 0, & \delta_{dg} K_{\mu\nu} &= 0, & \delta_{dg} \mathcal{K}_{\mu\nu} &= 0. \end{aligned} \quad (17)$$

One can check that, under the above infinitesimal (dual-)gauge symmetry transformations, the Lagrangian density  $\mathcal{L}_{(K)}$  transforms as:

$$\begin{aligned} \delta_{dg} \mathcal{L}_{(K)} &= - \mathcal{K}^{\mu\nu} \left[ \square \Sigma_{\mu\nu} + \partial_\mu (\partial^\eta \Sigma_{\nu\eta}) - \partial_\nu (\partial^\eta \Sigma_{\mu\eta}) \right], \\ \delta_g \mathcal{L}_{(K)} &= + K^{\mu\nu} \left[ \square \Omega_{\mu\nu} + \partial_\mu (\partial^\eta \Omega_{\nu\eta}) - \partial_\nu (\partial^\eta \Omega_{\mu\eta}) \right]. \end{aligned} \quad (18)$$

Thus, once again, the restrictions (13) have to be imposed for the (dual-)gauge invariance of  $\mathcal{L}_{(K)}$ . This is due to the self-duality invariance in the theory.

The generalization of the discrete symmetry transformations (15) can be written for the Lagrangian density  $\mathcal{L}_{(K)}$ , in terms of its fields, as

$$K_{\mu\nu} \longrightarrow \pm i \mathcal{K}_{\mu\nu}, \quad \mathcal{K}_{\mu\nu} \longrightarrow \pm i K_{\mu\nu}, \quad A_{\mu\nu\eta} \longrightarrow \pm \frac{i}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}. \quad (19)$$

It is interesting to point out that the Lagrangian density (16) can be further generalized by incorporating the Lorentz vector fields  $\phi_\mu^{(1)}$  and  $\phi_\mu^{(2)}$  as given below:

$$\begin{aligned} \mathcal{L}_{(\phi, K)} &= \frac{1}{2} \mathcal{K}^{\mu\nu} \mathcal{K}_{\mu\nu} - \mathcal{K}^{\mu\nu} \left( \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} + \frac{1}{2} [\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}] \right) \\ &+ K^{\mu\nu} \left( \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} [\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}] \right) - \frac{1}{2} K^{\mu\nu} K_{\mu\nu}, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathcal{L}_{(\phi, \bar{K})} &= \frac{1}{2} \bar{\mathcal{K}}^{\mu\nu} \bar{\mathcal{K}}_{\mu\nu} + \bar{\mathcal{K}}^{\mu\nu} \left( \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} - \frac{1}{2} [\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}] \right) \\ &- \bar{K}^{\mu\nu} \left( \partial^\eta A_{\eta\mu\nu} - \frac{1}{2} [\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}] \right) - \frac{1}{2} \bar{K}^{\mu\nu} \bar{K}_{\mu\nu}, \end{aligned} \quad (21)$$

where  $\bar{\mathcal{K}}_{\mu\nu}$  and  $\bar{K}_{\mu\nu}$  are the additional Nakanishi-Lautrup auxiliary fields that have been invoked for the most general form of the gauge-fixed Lagrangian densities. It should be mentioned here that we have freedom to add/subtract the 2-forms:  $F^{(2)} = d\Phi^{(1)} = \frac{1}{2}(dx^\mu \wedge dx^\nu)[\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}]$ ,  $\mathcal{F}^{(2)} = d\tilde{\Phi}^{(1)} = \frac{1}{2}(dx^\mu \wedge dx^\nu)[\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}]$  to the 2-forms  $*H^{(4)} = \frac{1}{4!}(dx^\mu \wedge dx^\nu) \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma}$  and  $\delta A^{(3)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu)(\partial^\eta A_{\eta\mu\nu})$  that are present in the 6D



Lagrangian density (16). The above coupled set of Lagrangian densities will be further generalized for the BRST analysis of the present theory in the forthcoming sections.

The equations of motion that emerge from the Lagrangian density  $\mathcal{L}_{(\phi,K)}$  are:

$$\begin{aligned} \mathcal{K}_{\mu\nu} &= \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} + \frac{1}{2} [\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}], & \square \phi_\mu^{(2)} - \partial_\mu (\partial \cdot \phi^{(2)}) &= 0, \\ K_{\mu\nu} &= \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} [\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}], & \square \phi_\mu^{(1)} - \partial_\mu (\partial \cdot \phi^{(1)}) &= 0, \quad \square A_{\mu\nu\eta} = 0, \\ \partial_\mu K_{\nu\eta} + \partial_\nu K_{\eta\mu} + \partial_\eta K_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \mathcal{K}^{\rho\sigma} &= 0, & \partial_\mu K^{\mu\nu} &= 0, \quad \square \mathcal{K}_{\mu\nu} = 0, \\ \partial_\mu \mathcal{K}_{\nu\eta} + \partial_\nu \mathcal{K}_{\eta\mu} + \partial_\eta \mathcal{K}_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa K^{\rho\sigma} &= 0, & \partial_\mu \mathcal{K}^{\mu\nu} &= 0, \quad \square K_{\mu\nu} = 0. \end{aligned} \quad (22)$$

Furthermore, the equations of motion, that are derived from the coupled Lagrangian density  $\mathcal{L}_{(\phi,\bar{K})}$ , are same as (22) except the following equations:

$$\begin{aligned} \bar{\mathcal{K}}_{\mu\nu} &= -\frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} + \frac{1}{2} [\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}], & \bar{K}_{\mu\nu} &= -\partial^\eta A_{\eta\mu\nu} + \frac{1}{2} [\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}], \\ \partial_\mu \bar{\mathcal{K}}_{\nu\eta} + \partial_\nu \bar{\mathcal{K}}_{\eta\mu} + \partial_\eta \bar{\mathcal{K}}_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \bar{\mathcal{K}}^{\rho\sigma} &= 0, & \partial_\mu \bar{K}^{\mu\nu} &= 0, \quad \square \bar{\mathcal{K}}_{\mu\nu} = 0, \\ \partial_\mu \bar{\mathcal{K}}_{\nu\eta} + \partial_\nu \bar{\mathcal{K}}_{\eta\mu} + \partial_\eta \bar{\mathcal{K}}_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \bar{K}^{\rho\sigma} &= 0, & \partial_\mu \bar{\mathcal{K}}^{\mu\nu} &= 0, \quad \square \bar{K}_{\mu\nu} = 0. \end{aligned} \quad (23)$$

We infer from the above equations that we have the following CF-type of restrictions:

$$K_{\mu\nu} + \bar{K}_{\mu\nu} = \partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}, \quad \mathcal{K}_{\mu\nu} + \bar{\mathcal{K}}_{\mu\nu} = \partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}. \quad (24)$$

The above CF-type of conditions are responsible for the equivalence of the Lagrangian density (20) and (21) which can be *checked explicitly* (modulo some total spacetime derivatives). This is the reason that we call these Lagrangian densities as the coupled and equivalent Lagrangian densities for our present 6D Abelian 3-form gauge theory.

The discrete symmetry transformations (19) can be further generalized for the coupled Lagrangian densities (20) and (21) as given below:

$$\begin{aligned} K_{\mu\nu} &\longrightarrow \pm i \mathcal{K}_{\mu\nu}, & \mathcal{K}_{\mu\nu} &\longrightarrow \pm i K_{\mu\nu}, & A_{\mu\nu\eta} &\longrightarrow \pm \frac{i}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}, \\ \phi_\mu^{(1)} &\longrightarrow \pm i \phi_\mu^{(2)}, & \phi_\mu^{(2)} &\longrightarrow \pm i \phi_\mu^{(1)}, \end{aligned} \quad (25)$$

$$\begin{aligned} \bar{K}_{\mu\nu} &\longrightarrow \pm i \bar{\mathcal{K}}_{\mu\nu}, & \bar{\mathcal{K}}_{\mu\nu} &\longrightarrow \pm i \bar{K}_{\mu\nu}, & A_{\mu\nu\eta} &\longrightarrow \pm \frac{i}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}, \\ \phi_\mu^{(1)} &\longrightarrow \pm i \phi_\mu^{(2)}, & \phi_\mu^{(2)} &\longrightarrow \pm i \phi_\mu^{(1)}. \end{aligned} \quad (26)$$

The above transformations are the *symmetry* transformations for the Lagrangian densities (20) and (21). Under the following continuous (dual-)gauge transformations

$$\begin{aligned} \delta_{dg} A_{\mu\nu\eta} &= \frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \Sigma^{\rho\sigma}, & \delta_{dg} (\partial^\eta A_{\eta\mu\nu}) &= 0, & \delta_{dg} K_{\mu\nu} &= 0, \\ \delta_{dg} \mathcal{K}_{\mu\nu} &= 0, & \delta_{dg} \phi_\mu^{(1)} &= 0, & \delta_{dg} \phi_\mu^{(2)} &= \partial^\eta \Sigma_{\eta\mu} + \partial_\mu \chi, \\ \delta_{dg} A_{\mu\nu\eta} &= \partial_\mu \Omega_{\nu\eta} + \partial_\nu \Omega_{\eta\mu} + \partial_\eta \Omega_{\mu\nu}, & \delta_{dg} (H_{\mu\nu\eta\kappa}) &= 0, & \delta_{dg} K_{\mu\nu} &= 0, \\ \delta_{dg} \mathcal{K}_{\mu\nu} &= 0, & \delta_{dg} \phi_\mu^{(2)} &= 0, & \delta_{dg} \phi_\mu^{(1)} &= \partial^\eta \Omega_{\eta\mu} + \partial_\mu \zeta, \end{aligned} \quad (27)$$

the coupled Lagrangian densities (20) and (21) transform as:

$$\begin{aligned}
\delta_{dg}\mathcal{L}_{(\phi,K)} &= -\mathcal{K}^{\mu\nu}\left[\Box\Sigma_{\mu\nu} + \partial_\mu(\partial^\eta\Sigma_{\nu\eta}) - \partial_\nu(\partial^\eta\Sigma_{\mu\eta})\right] + \frac{1}{2}(\partial_\eta\Sigma^{\eta\nu})\left[\Box\phi_\nu^{(2)} - \partial_\nu(\partial\cdot\phi^{(2)})\right], \\
\delta_{dg}\mathcal{L}_{(\phi,\bar{K})} &= \bar{\mathcal{K}}^{\mu\nu}\left[\Box\Sigma_{\mu\nu} + \partial_\mu(\partial^\eta\Sigma_{\nu\eta}) - \partial_\nu(\partial^\eta\Sigma_{\mu\eta})\right] + \frac{1}{2}(\partial_\eta\Sigma^{\eta\nu})\left[\Box\phi_\nu^{(2)} - \partial_\nu(\partial\cdot\phi^{(2)})\right], \\
\delta_g\mathcal{L}_{(\phi,K)} &= K^{\mu\nu}\left[\Box\Omega_{\mu\nu} + \partial_\mu(\partial^\eta\Omega_{\nu\eta}) - \partial_\nu(\partial^\eta\Omega_{\mu\eta})\right] - \frac{1}{2}(\partial_\eta\Omega^{\eta\nu})\left[\Box\phi_\nu^{(1)} - \partial_\nu(\partial\cdot\phi^{(1)})\right], \\
\delta_g\mathcal{L}_{(\phi,\bar{K})} &= -\bar{K}^{\mu\nu}\left[\Box\Omega_{\mu\nu} + \partial_\mu(\partial^\eta\Omega_{\nu\eta}) - \partial_\nu(\partial^\eta\Omega_{\mu\eta})\right] - \frac{1}{2}(\partial_\eta\Omega^{\eta\nu})\left[\Box\phi_\nu^{(1)} - \partial_\nu(\partial\cdot\phi^{(1)})\right]. \quad (28)
\end{aligned}$$

It should be noted that the 2-forms  $F^{(2)} = d\Phi^{(1)}$  and  $\mathcal{F}^{(2)} = d\tilde{\Phi}^{(1)}$ , present in the Lagrangian densities (20) and (21), permit us to have the vector  $U(1)$  gauge transformations  $\Phi^{(1)} \rightarrow \Phi'^{(1)} = \Phi^{(1)} + d\zeta^{(0)}$  and  $\tilde{\Phi}^{(1)} \rightarrow \tilde{\Phi}'^{(1)} = \tilde{\Phi}^{(1)} + d\chi^{(0)}$ . As a consequence, in the (dual-)gauge transformations (27), we have included the (dual-)gauge parameters  $\chi$  and  $\zeta$  corresponding to the zero-forms  $\chi^{(0)}$  and  $\zeta^{(0)}$ . It can be checked that, under the following conditions:

$$\begin{aligned}
\Box\Omega_{\mu\nu} + \partial_\mu(\partial^\eta\Omega_{\nu\eta}) - \partial_\nu(\partial^\eta\Omega_{\mu\eta}) &= 0, & \Box\phi_\mu^{(1)} - \partial_\mu(\partial\cdot\phi^{(1)}) &= 0, \\
\Box\Sigma_{\mu\nu} + \partial_\mu(\partial^\eta\Sigma_{\nu\eta}) - \partial_\nu(\partial^\eta\Sigma_{\mu\eta}) &= 0, & \Box\phi_\mu^{(2)} - \partial_\mu(\partial\cdot\phi^{(2)}) &= 0, \quad (29)
\end{aligned}$$

the Lagrangian densities  $\mathcal{L}_{(\phi,K)}$  and  $\mathcal{L}_{(\phi,\bar{K})}$  remain invariant. In the next section, we shall see that these restrictions would emerge naturally and would *not* be required to be imposed on the theory (from outside) within the framework of BRST formalism.

A close look at the 2D Abelian 1-form gauge theory, 4D Abelian 2-form gauge theory and 6D Abelian 3-form gauge theory allows us to generalize our results to an arbitrary Abelian  $p$ -form gauge theory. We note that such general  $D$ -dimensional Abelian theories would have dual-gauge symmetry [and corresponding (anti-)dual-BRST symmetries] *whenever the condition:  $*dA^{(p)} = \delta A^{(p)}$  is satisfied*. This happens only when the dimension of the spacetime turns out to be exactly equal to  $[(2p-1)+1]$  (i.e.  $D=2p$ ). In a very concise manner, we discuss the self-duality condition for a general Abelian  $p$ -form gauge field in  $D=2p$  dimensions of spacetime for the gauge-fixed Lagrangian density in our Appendix D (which is valid for the simple case of our choice of gauge-fixing being the Feynman gauge).

### 3 (Anti-)BRST symmetries: conserved charges

The most general forms of the gauge-fixed coupled (but equivalent) Lagrangian densities (20) and (21) can be obtained by incorporating Faddeev-Popov ghost terms as follows [22]

$$\begin{aligned}
\mathcal{L}_{(b)} &= \frac{1}{2}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - \mathcal{K}^{\mu\nu}\left(\frac{1}{4!}\varepsilon_{\mu\nu\eta\kappa\rho\sigma}H^{\eta\kappa\rho\sigma} + \frac{1}{2}[\partial_\mu\phi_\nu^{(2)} - \partial_\nu\phi_\mu^{(2)}]\right) - \frac{1}{2}K^{\mu\nu}K_{\mu\nu} - BB_2 \\
&+ K^{\mu\nu}\left(\partial^\eta A_{\eta\mu\nu} + \frac{1}{2}[\partial_\mu\phi_\nu^{(1)} - \partial_\nu\phi_\mu^{(1)}]\right) + \left(\partial_\mu\bar{C}_{\nu\eta} + \partial_\nu\bar{C}_{\eta\mu} + \partial_\eta\bar{C}_{\mu\nu}\right)(\partial^\mu C^{\nu\eta}) \\
&+ (\partial\cdot\phi^{(1)})B_1 - (\partial\cdot\phi^{(2)})B_3 - \frac{1}{2}B_1^2 + \frac{1}{2}B_3^2 + (\partial\cdot\bar{\beta})B - (\partial_\mu\bar{\beta}_\nu - \partial_\nu\bar{\beta}_\mu)(\partial^\mu\beta^\nu) \\
&- (\partial\cdot\beta)B_2 - 2\bar{F}^\mu f_\mu + (\partial_\mu\bar{C}^{\mu\nu} + \partial^\nu\bar{C}_1)f_\nu - (\partial_\mu C^{\mu\nu} + \partial^\nu C_1)\bar{F}_\nu - \partial_\mu\bar{C}_2\partial^\mu C_2, \quad (30)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{(\bar{b})} = & \frac{1}{2} \bar{\mathcal{K}}_{\mu\nu} \bar{\mathcal{K}}^{\mu\nu} + \bar{\mathcal{K}}^{\mu\nu} \left( \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} - \frac{1}{2} [\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}] \right) - \frac{1}{2} \bar{K}^{\mu\nu} \bar{K}_{\mu\nu} - B B_2 \\
& - \bar{K}^{\mu\nu} \left( \partial^\eta A_{\eta\mu\nu} - \frac{1}{2} [\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}] \right) + \left( \partial_\mu \bar{C}_{\nu\eta} + \partial_\nu \bar{C}_{\eta\mu} + \partial_\eta \bar{C}_{\mu\nu} \right) (\partial^\mu C^{\nu\eta}) \\
& + (\partial \cdot \phi^{(1)}) B_1 - (\partial \cdot \phi^{(2)}) B_3 - \frac{1}{2} B_1^2 + \frac{1}{2} B_3^2 + (\partial \cdot \bar{\beta}) B - (\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu) (\partial^\mu \beta^\nu) \\
& - (\partial \cdot \beta) B_2 - 2 \bar{f}^\mu F_\mu - (\partial_\mu \bar{C}^{\mu\nu} - \partial^\nu \bar{C}_1) F_\nu + (\partial_\mu C^{\mu\nu} - \partial^\nu C_1) \bar{f}_\nu - \partial_\mu \bar{C}_2 \partial^\mu C_2, \quad (31)
\end{aligned}$$

where the fermionic antisymmetric tensor (anti-)ghost fields  $(\bar{C}_{\mu\nu})C_{\mu\nu}$  [with ghost number equal to  $(-1)+1$ ], the bosonic Lorentz vector (anti-)ghost fields  $(\bar{\beta}_\mu)\beta_\mu$  [with ghost number  $(-2)+2$ ], the Lorentz scalar fermionic (anti-)ghost fields  $(\bar{C}_2)C_2$  [with ghost number  $(-3)+3$ ] are required for the validity of unitarity in the theory. Furthermore, we have fermionic auxiliary (anti-)ghost fields  $(\bar{F}_\mu)F_\mu$  and  $(\bar{f}_\mu)f_\mu$  in the theory together with the (anti-)ghost fields  $(\bar{C}_1)C_1$ . All these fields have ghost number equal to  $(-1)+1$ . We have auxiliary fields  $B, B_1, B_2, B_3$  also in our complete theory which are used for the linearizations.

Our present coupled and equivalent Lagrangian densities (30) and (31) differ slightly from such Lagrangian densities in [22]. This is due to the fact that there are extra pieces in (30) and (31) that were absent in the corresponding Lagrangian densities in [22]. These terms are  $[-(\partial \cdot \phi^{(2)})B_3 + (1/2)B_3^2]$ ,  $[-2\bar{F}^\mu f_\mu]$  and  $[-2\bar{f}^\mu F_\mu]$  that are present in (30) and (31). The term  $[-(\partial \cdot \phi^{(2)})B_3 + (1/2)B_3^2]$  is required for the gauge-fixing of the vector field  $\phi_\mu^{(2)}$  and other two terms  $[-2\bar{F}^\mu f_\mu]$  and  $[-2\bar{f}^\mu F_\mu]$  are required so that the CF-type conditions (24) and (43) (see below) could be derived from (30) and (31). We discuss more about these issues in our Appendix A and establish the root-cause of the difference.

We can check that, under the following off-shell nilpotent ( $s_b^2 = 0$ ) supersymmetric type BRST symmetry transformations ( $s_b$ ) (see, e.g., [21,22] for details)

$$\begin{aligned}
s_b A_{\mu\nu\eta} &= \partial_\mu C_{\nu\eta} + \partial_\nu C_{\eta\mu} + \partial_\eta C_{\mu\nu}, & s_b C_{\mu\nu} &= \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, & s_b \bar{C}_{\mu\nu} &= K_{\mu\nu}, \\
s_b \bar{K}_{\mu\nu} &= \partial_\mu f_\nu - \partial_\nu f_\mu, & s_b \bar{\beta}_\mu &= \bar{F}_\mu, & s_b \beta_\mu &= \partial_\mu C_2, & s_b F_\mu &= -\partial_\mu B, \\
s_b \bar{C}_2 &= B_2, & s_b C_1 &= -B, & s_b \bar{C}_1 &= B_1, & s_b \phi_\mu^{(1)} &= f_\mu, & s_b \bar{f}_\mu &= \partial_\mu B_1, \\
s_b [C_2, f_\mu, \bar{F}_\mu, B, B_1, B_2, B_3, \phi_\mu^{(2)}, K_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}, H_{\mu\nu\eta\kappa}] &= 0, \quad (32)
\end{aligned}$$

the Lagrangian density  $\mathcal{L}_{(b)}$  transforms to a total spacetime derivative as given below

$$\begin{aligned}
s_b \mathcal{L}_{(b)} = & \partial_\mu \left[ (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) K_{\nu\eta} + K^{\mu\nu} f_\nu - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{F}_\nu \right. \\
& \left. + B_1 f^\mu + B \bar{F}^\mu - B_2 (\partial^\mu C_2) \right]. \quad (33)
\end{aligned}$$

Hence, the action integral  $S = \int d^6x \mathcal{L}_{(b)}$  would remain invariant for the physically well defined fields of the theory which vanish off at infinity (due to Gauss's divergence theorem).

Like the above BRST symmetry transformations, the Lagrangian density  $\mathcal{L}_{(\bar{b})}$  respects an off-shell nilpotent ( $s_{ab}^2 = 0$ ) anti-BRST symmetry transformations ( $s_{ab}$ ):

$$\begin{aligned}
s_{ab} A_{\mu\nu\eta} &= \partial_\mu \bar{C}_{\nu\eta} + \partial_\nu \bar{C}_{\eta\mu} + \partial_\eta \bar{C}_{\mu\nu}, & s_{ab} \bar{C}_{\mu\nu} &= \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, & s_{ab} C_{\mu\nu} &= \bar{K}_{\mu\nu}, \\
s_{ab} K_{\mu\nu} &= \partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu, & s_{ab} \beta_\mu &= F_\mu, & s_{ab} \bar{\beta}_\mu &= \partial_\mu \bar{C}_2, & s_{ab} \bar{F}_\mu &= -\partial_\mu B_2, \\
s_{ab} C_2 &= B, & s_{ab} f_\mu &= -\partial_\mu B_1, & s_{ab} C_1 &= -B_1, & s_{ab} \bar{C}_1 &= -B_2, & s_{ab} \phi_\mu^{(1)} &= \bar{f}_\mu, \\
s_{ab} [\bar{C}_2, \bar{f}_\mu, F_\mu, B, B_1, B_2, B_3, \phi_\mu^{(2)}, \bar{K}_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}, H_{\mu\nu\eta\kappa}] &= 0. \quad (34)
\end{aligned}$$

It can be checked that the Lagrangian density  $\mathcal{L}_{(\bar{b})}$  transforms, under the above transformations (34), to a spacetime total derivative as given below

$$\begin{aligned} s_{ab}\mathcal{L}_{(\bar{b})} &= \partial_\mu \left[ -(\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) \bar{K}_{\nu\eta} + \bar{K}^{\mu\nu} \bar{f}_\nu - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) F_\nu \right. \\ &\quad \left. + B_1 \bar{f}^\mu - B_2 F^\mu + B (\partial^\mu \bar{C}_2) \right]. \end{aligned} \quad (35)$$

Thus, the action integral in 6D spacetime ( $S = \int d^6x \mathcal{L}_{(\bar{b})}$ ) remains invariant for the physically well-defined fields of the theory which fall off rapidly at infinity. To be explicit, it is Gauss's divergence theorem that implies that all fields of the r.h.s. of (35) will be evaluated at infinity and, because of the physical arguments, these fields would go to zero.

According to Noether's theorem, the invariance of an action under the continuous symmetry transformations, leads to the existence of conserved currents. From the action principle, it turns out that these Noether currents are conserved because of the validity of Euler-Lagrange equations of motion (that also ensue from the least action principle). Ultimately, we have the following expressions for the conserved currents  $J_{(ab)}^\mu$  and  $J_{(b)}^\mu$  for the (anti-)BRST symmetry invariance of the Lagrangian densities  $\mathcal{L}_{(\bar{b})}$ , and  $\mathcal{L}_{(b)}$ , namely;

$$\begin{aligned} J_{(ab)}^\mu &= \frac{1}{2!} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} (\partial_\nu \bar{C}_{\eta\kappa}) \bar{K}_{\lambda\rho} - (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) \bar{K}_{\nu\eta} + \bar{K}^{\mu\nu} \bar{f}_\nu \\ &\quad - B_2 F^\mu + B (\partial^\mu \bar{C}_2) + (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) (\partial_\nu \bar{\beta}_\eta - \partial_\eta \bar{\beta}_\nu) \\ &\quad - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) (\partial_\nu \bar{C}_2) - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) F_\nu + B_1 \bar{f}^\mu, \end{aligned} \quad (36)$$

$$\begin{aligned} J_{(b)}^\mu &= -\frac{1}{2!} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} (\partial_\nu C_{\eta\kappa}) K_{\lambda\rho} + (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) K_{\nu\eta} + K^{\mu\nu} f_\nu \\ &\quad + B_1 f^\mu - B_2 (\partial^\mu C_2) - (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) (\partial_\nu \beta_\eta - \partial_\eta \beta_\nu) \\ &\quad - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) (\partial_\nu C_2) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{F}_\nu + B \bar{F}^\mu. \end{aligned} \quad (37)$$

The conservation law  $[\partial_\mu J_{((a)b)}^\mu = 0]$  for the (anti-)BRST currents could be proven by exploiting the following Euler-Lagrange equations of motion from  $\mathcal{L}_{(b)}$ , namely;

$$\begin{aligned} K_{\mu\nu} &= \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}), \quad \square \phi_\mu^{(1)} + \partial_\mu (\partial \cdot \phi^{(1)}) = 0, \quad \square C_1 = 0, \\ \mathcal{K}_{\mu\nu} &= \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} H^{\eta\kappa\lambda\rho} + \frac{1}{2} (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}), \quad \square \phi_\mu^{(2)} + \partial_\mu (\partial \cdot \phi^{(2)}) = 0, \\ \frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu K^{\nu\eta}) &= \partial_\kappa \mathcal{K}_{\lambda\rho} + \partial_\lambda \mathcal{K}_{\rho\kappa} + \partial_\rho \mathcal{K}_{\kappa\lambda}, \quad \square \bar{C}_1 = 0, \quad \square C_2 = 0, \\ \frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu \mathcal{K}^{\nu\eta}) &= \partial_\kappa K_{\lambda\rho} + \partial_\lambda K_{\rho\kappa} + \partial_\rho K_{\kappa\lambda}, \quad \square \bar{C}_2 = 0, \quad \square B_1 = 0, \\ \square C_{\mu\nu} - \frac{3}{2} (\partial_\mu f_\nu - \partial_\nu f_\mu) &= 0, \quad \square \bar{C}_{\mu\nu} - \frac{3}{2} (\partial_\mu \bar{F}_\nu - \partial_\nu \bar{F}_\mu) = 0, \quad \square B_3 = 0, \\ \partial_\mu K^{\mu\nu} + \partial^\nu B_1 &= 0, \quad \partial_\mu \mathcal{K}^{\mu\nu} + \partial^\nu B_3 = 0, \quad B_1 = (\partial \cdot \phi^{(1)}), \quad \square \bar{F}_\mu = 0, \\ B &= -(\partial \cdot \beta), \quad B_2 = (\partial \cdot \bar{\beta}), \quad B_3 = (\partial \cdot \phi^{(2)}), \quad \partial \cdot \bar{F} = 0, \quad \square f_\mu = 0, \\ \partial_\mu C^{\mu\nu} + \partial^\nu C_1 - 2f^\nu &= 0, \quad \partial_\mu \bar{C}^{\mu\nu} + \partial^\nu \bar{C}_1 - 2\bar{F}^\nu = 0, \quad \partial \cdot f = 0, \\ \square K_{\mu\nu} &= 0, \quad \square \mathcal{K}_{\mu\nu} = 0, \quad \square \beta_\mu = 0, \quad \square \bar{\beta}_\mu = 0, \end{aligned} \quad (38)$$

and the ones that emerge from the Lagrangian density  $\mathcal{L}_{(\bar{b})}$  are:

$$\begin{aligned}
\bar{K}_{\mu\nu} &= -\partial^\eta A_{\eta\mu\nu} + \frac{1}{2} (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}), \quad \square \phi_\mu^{(1)} + \partial_\mu (\partial \cdot \phi^{(1)}) = 0, \quad \square C_1 = 0, \\
\bar{\mathcal{K}}_{\mu\nu} &= -\frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} H^{\eta\kappa\lambda\rho} + \frac{1}{2} (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}), \quad \square \phi_\mu^{(2)} + \partial_\mu (\partial \cdot \phi^{(2)}) = 0, \\
\frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu \bar{K}^{\nu\eta}) &= \partial_\kappa \bar{\mathcal{K}}_{\lambda\rho} + \partial_\lambda \bar{\mathcal{K}}_{\rho\kappa} + \partial_\rho \bar{\mathcal{K}}_{\kappa\lambda}, \quad \square \bar{C}_1 = 0, \quad \square C_2 = 0, \\
\frac{1}{2!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu \bar{\mathcal{K}}^{\nu\eta}) &= \partial_\kappa \bar{K}_{\lambda\rho} + \partial_\lambda \bar{K}_{\rho\kappa} + \partial_\rho \bar{K}_{\kappa\lambda}, \quad \square \bar{C}_2 = 0, \quad \square B_1 = 0, \\
\square C_{\mu\nu} + \frac{3}{2} (\partial_\mu F_\nu - \partial_\nu F_\mu) &= 0, \quad \square \bar{C}_{\mu\nu} + \frac{3}{2} (\partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu) = 0, \quad \square B_3 = 0, \\
\partial_\mu \bar{K}^{\mu\nu} + \partial^\nu B_1 &= 0, \quad \partial_\mu \bar{\mathcal{K}}^{\mu\nu} + \partial^\nu B_3 = 0, \quad B_2 = (\partial \cdot \bar{\beta}), \quad \square F_\mu = 0, \\
B &= -(\partial \cdot \beta), \quad B_1 = (\partial \cdot \phi^{(1)}), \quad B_3 = (\partial \cdot \phi^{(2)}), \quad \partial \cdot \bar{f} = 0, \quad \square \bar{f}_\mu = 0, \\
\partial_\mu C^{\mu\nu} - \partial^\nu C_1 + 2F^\nu &= 0, \quad \partial_\mu \bar{C}^{\mu\nu} - \partial^\nu \bar{C}_1 + 2\bar{f}^\nu = 0, \quad \partial \cdot F = 0, \\
\square K_{\mu\nu} &= 0, \quad \square \mathcal{K}_{\mu\nu} = 0, \quad \square \beta_\mu = 0, \quad \square \bar{\beta}_\mu = 0.
\end{aligned} \tag{39}$$

The above conserved currents  $J_{((a)b)}^\mu$  lead to the derivation of the following off-shell nilpotent ( $Q_{(a)b}^2 = 0$ ) and conserved ( $\dot{Q}_{(a)b} = 0$ ) (anti-)BRST charges  $Q_{(a)b}$ :

$$\begin{aligned}
Q_{ab} &= \int d^5x \left[ \frac{1}{2!} \varepsilon^{0ijklm} (\partial_i \bar{C}_{jk}) \bar{\mathcal{K}}_{lm} - (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) \bar{K}_{\nu\eta} + \bar{K}^{0i} \bar{f}_i \right. \\
&\quad + B_1 \bar{f}^0 + B \dot{\bar{C}}_2 + (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) (\partial_\nu \bar{\beta}_\eta - \partial_\eta \bar{\beta}_\nu) - B_2 F^0 \\
&\quad \left. - (\partial^0 \beta^i - \partial^i \beta^0) (\partial_i \bar{C}_2) - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) F_i \right],
\end{aligned} \tag{40}$$

$$\begin{aligned}
Q_b &= \int d^5x \left[ -\frac{1}{2!} \varepsilon^{0ijklm} (\partial_i C_{jk}) \mathcal{K}_{lm} + (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) K_{\nu\eta} + K^{0i} f_i \right. \\
&\quad + B_1 f^0 - B_2 \dot{C}_2 - (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) (\partial_\nu \beta_\eta - \partial_\eta \beta_\nu) + B \bar{F}^0 \\
&\quad \left. - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i C_2) - (\partial^0 \beta^i - \partial^i \beta^0) \bar{F}_i \right].
\end{aligned} \tag{41}$$

The above conserved charges  $Q_{(a)b}$  are the generators of the local, continuous and infinitesimal (anti-)BRST symmetry transformations, as it can be checked that

$$s_{(a)b} \Psi = \pm i [\Psi, Q_{(a)b}]_\pm, \tag{42}$$

where the (+)– signs, present as the subscripts on the square bracket, correspond to the (anti)commutators for the generic field  $\Psi$  being (fermionic) bosonic in nature. Similarly, the ( $\pm$ ) signs, in front of the square bracket, are chosen appropriately (see, e.g., [27]). Furthermore, one has to use the appropriate canonical brackets (that are derived from the Lagrangian densities  $\mathcal{L}_{(b)}$  and  $\mathcal{L}_{(\bar{b})}$ ) in the evaluations of the above (anti)commutators.

We would like to mention that there are *four* CF-type restrictions [21,22] on the theory which have been derived by exploiting the theoretical potential and power of superfield

approach to BRST formalism [28,29]. All of these (that are bosonic and fermionic in nature) can also be derived by exploiting the equations of motion (38) and (39). The fermionic CF-type restrictions [amongst the fermionic (anti-)ghost fields] were originally derived by exploiting the superfield technique [21]. These useful and interesting restrictions are

$$f_\mu + F_\mu = \partial_\mu C_1, \quad \bar{f}_\mu + \bar{F}_\mu = \partial_\mu \bar{C}_1. \quad (43)$$

Concentrating on the last but one rows of (38) and (39), it is evident that these Euler-Lagrange equations of motion produce the above CF-type restrictions (43). Exploiting the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations (32) and (34), it can be checked that the restrictions in (24) and (43) are (anti-)BRST invariant quantities and, hence, they are physical restrictions on the theory. We re-emphasize that the coupled Lagrangian densities (30) and (31) produce all the CF-type restrictions (24) and (43) as a set of off-shoots from the equations of motion (38) and (39).

The roles of the (anti-)BRST invariant CF-type restrictions (24) and (43) are two folds. First, these allow us to obtain a set of two coupled (but equivalent) Lagrangian densities (30) and (31) for the theory. Second, we observe that the above nilpotent (anti-)BRST symmetry transformations ( $s_{(a)b}$ ) obey perfect absolute anticommutativity property  $\{s_b, s_{ab}\} = 0$  for all the fields of the theory except the following:

$$\{s_b, s_{ab}\} A_{\mu\nu\eta} \neq 0, \quad \{s_b, s_{ab}\} C_{\mu\nu} \neq 0, \quad \{s_b, s_{ab}\} \bar{C}_{\mu\nu} \neq 0. \quad (44)$$

If we compute the above anticommutators, in a straightforward manner, they turn out to be non-zero. However, the above fields ( $A_{\mu\nu\eta}, C_{\mu\nu}, \bar{C}_{\mu\nu}$ ) also respect the absolute anticommutativity property on the constrained hypersurface (in the 6D spacetime manifold) where the CF-type restrictions (24) and (43) are valid. For instance, only due to (24), we have  $\{s_b, s_{ab}\} A_{\mu\nu\eta} = 0$ . In addition, the validity of  $\{s_b, s_{ab}\} C_{\mu\nu} = 0$  and  $\{s_b, s_{ab}\} \bar{C}_{\mu\nu} = 0$  is true only when (43) is satisfied. We re-emphasize, once again, that restrictions (24) and (43) have been derived due to the superfield approach to Abelian 3-form gauge theory [21] and, to the best of our knowledge, they cannot be derived by using any other method. However, it is straightforward to note that (24) and (43) can be derived from the equations of motion (38) and (39) as well. The interesting point to be emphasized is that the coupled Lagrangian densities (30) and (31), which produce (38) and (39), have been derived from the knowledge of  $s_{(a)b}$ . However, the off-shell nilpotent ( $s_{(a)b}^2 = 0$ ) and absolutely anticommuting ( $s_b s_{ab} + s_{ab} s_b = 0$ ) symmetries  $s_{(a)b}$  have been derived from the superfield formalism (see, e.g., [21] for details). Hence, it is the superfield formalism which is more basic.

We wrap up this section with some remarks. First, the absolute anticommutativity of the (anti-)BRST charges  $Q_{(a)b}$  is satisfied only on the constrained hypersurface defined by the field equations corresponding to the CF-type restrictions (24) and (43). Second, the physicality criteria:  $Q_{(a)b}|phys\rangle = 0$  leads to the annihilation of the physical states by the operator form of the first-class constraints of the theory (cf. Sec. VII for details). Third, both the Lagrangian densities  $\mathcal{L}_{(b)}$  and  $\mathcal{L}_{(\bar{b})}$  respect both the off-shell nilpotent (anti-)BRST symmetry transformations on the hypersurface (in the 6D spacetime manifold) which is described by the CF-type field equations. This statement can be succinctly expressed in

the following mathematical form [cf. (24), (43)], namely;

$$\begin{aligned}
s_{ab}\mathcal{L}_{(b)} &= \partial_\mu \left[ (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) K_{\nu\eta} + A^{\mu\nu\eta} (\partial_\nu \bar{f}_\eta - \partial_\eta \bar{f}_\nu) + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) f_\nu \right. \\
&\quad - \bar{K}^{\mu\nu} \bar{F}_\nu + \bar{C}^{\mu\nu} (\partial_\nu B_1) - C^{\mu\nu} (\partial_\nu B_2) + B (\partial^\mu \bar{C}_2) - B_2 F^\mu + B_1 \bar{f}^\mu \Big] \\
&\quad - (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) \partial_\mu \left[ \bar{K}_{\nu\eta} + K_{\nu\eta} - (\partial_\nu \phi_\eta^{(1)} - \partial_\eta \phi_\nu^{(1)}) \right] \\
&\quad - \left[ \bar{K}_{\mu\nu} + K_{\mu\nu} - (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}) \right] (\partial^\mu \bar{f}^\nu) + \bar{K}^{\mu\nu} \partial_\mu \left[ \bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1 \right] \\
&\quad - (\partial^\mu B_1) \left[ \bar{f}_\mu + \bar{F}_\mu - \partial_\mu \bar{C}_1 \right] + (\partial^\mu B_2) \left[ f_\mu + F_\mu - \partial_\mu C_1 \right] \\
&\quad - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \left[ f_\nu + F_\nu - \partial_\nu C_1 \right], \tag{45}
\end{aligned}$$

$$\begin{aligned}
s_b\mathcal{L}_{(\bar{b})} &= -\partial_\mu \left[ (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \bar{K}_{\nu\eta} + A^{\mu\nu\eta} (\partial_\nu f_\eta - \partial_\eta f_\nu) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{f}_\nu \right. \\
&\quad + K^{\mu\nu} F_\nu + \bar{C}^{\mu\nu} (\partial_\nu B) + C^{\mu\nu} (\partial_\nu B_1) + B_2 (\partial^\mu C_2) - B \bar{F}^\mu - B_1 f^\mu \Big] \\
&\quad + (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \partial_\mu \left[ \bar{K}_{\nu\eta} + K_{\nu\eta} - (\partial_\nu \phi_\eta^{(1)} - \partial_\eta \phi_\nu^{(1)}) \right] \\
&\quad - \left[ \bar{K}_{\mu\nu} + K_{\mu\nu} - (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}) \right] (\partial^\mu f^\nu) + K^{\mu\nu} \partial_\mu \left[ f_\nu + F_\nu - \partial_\nu C_1 \right] \\
&\quad - (\partial^\mu B) \left[ \bar{f}_\mu + \bar{F}_\mu - \partial_\mu \bar{C}_1 \right] - (\partial^\mu B_1) \left[ f_\mu + F_\mu - \partial_\mu C_1 \right] \\
&\quad - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \partial_\mu \left[ \bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1 \right]. \tag{46}
\end{aligned}$$

Thus, we note that the coupled Lagrangian densities (30) and (31) are equivalent as far as the nilpotent (anti-)BRST symmetry transformations on the constrained hypersurface, defined by the CF-type field equations (24) and (43), are concerned. Finally, under the off-shell nilpotent (anti-)BRST transformations, it is interesting to point out that the curvature tensor  $H_{\mu\nu\eta\kappa}$  (owing its origin to the exterior derivative) remains invariant.

It appears that the (anti-)BRST symmetries provide a physical realization of the exterior derivative. However, the absolute anticommutativity property ( $s_b s_{ab} + s_{ab} s_b = 0$ ) of the nilpotent (anti-)BRST symmetry transformations imply that *only one* of them could be identified with the exterior derivative of differential geometry because the BRST and anti-BRST symmetry transformations are linearly independent of each-other. In fact, as it turns out, it is the BRST symmetry transformations  $s_b$  (and corresponding charge  $Q_b$ ) that provide the physical realization of the exterior derivative. An extensive discussion on these issues could be found in our Sec. VII [see, equations (75) and (79)] where we have shown the explicit mappings between the cohomological operators of differential geometry and conserved charges of our present gauge theory. Of course, there is another physical quantity which also provides a realization of the exterior derivative but that happens only in the six  $(5+1)$ -dimensions of spacetime where we obtain the off-shell nilpotent (anti-)co-BRST symmetry transformations and their corresponding conserved and nilpotent charges. These nilpotent charges, in fact, provide physical realizations of the (co-)exterior derivatives of differential geometry (see, e.g., Sec. VII for more details).

## 4 (Anti-)co-BRST symmetries: conserved charges

It is interesting to note that the Lagrangian density  $\mathcal{L}_{(b)}$  remains quasi-invariant under the following off-shell nilpotent ( $s_d^2 = 0$ ) co-BRST/dual-BRST symmetry transformations ( $s_d$ ):

$$\begin{aligned} s_d A_{\mu\nu\eta} &= \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \bar{C}^{\rho\sigma}, & s_d \bar{C}_{\mu\nu} &= \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, & s_d \bar{\beta}_\mu &= \partial_\mu \bar{C}_2, \\ s_d \bar{C}_1 &= B_2, & s_d \beta_\mu &= -f_\mu, & s_d C_1 &= B_3, & s_d \phi_\mu^{(2)} &= \bar{F}_\mu, & s_d C_2 &= B, \\ s_d C_{\mu\nu} &= K_{\mu\nu}, & s_d \bar{f}_\mu &= \partial_\mu B_2, & s_d \bar{K}_{\mu\nu} &= \partial_\mu \bar{F}_\nu - \partial_\nu \bar{F}_\mu, & s_d F_\mu &= \partial_\mu B_3, \\ s_d [\partial^\eta A_{\eta\mu\nu}, \phi_\mu^{(1)}, K_{\mu\nu}, \bar{K}_{\mu\nu}, K_{\mu\nu}, B, B_1, B_2, B_3, \bar{C}_2, f_\mu, \bar{F}_\mu] &= 0, \end{aligned} \quad (47)$$

because the Lagrangian density  $\mathcal{L}_{(b)}$  transforms to a total spacetime derivative as

$$\begin{aligned} s_d \mathcal{L}_{(b)} &= -\partial_\mu \left[ (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) K_{\nu\eta} - B (\partial^\mu \bar{C}_2) - B_2 f^\mu + K^{\mu\nu} \bar{F}_\nu \right. \\ &\quad \left. + B_3 \bar{F}^\mu - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) f_\nu \right]. \end{aligned} \quad (48)$$

One of the decisive features of the dual-BRST symmetry is the invariance of the total gauge-fixing term [ i.e.  $s_d(\partial^\mu A_{\mu\nu\eta}) = 0$ ,  $s_d K_{\mu\nu} = 0$ ,  $s_d \phi_\mu^{(1)} = 0$ ] which owes its origin to the co-exterior derivative  $\delta = - * d *$  as we have  $\delta A^{(3)} = (1/2!)(dx^\mu \wedge dx^\nu)(\partial^\eta A_{\eta\mu\nu})$ .

Analogous to the transformations (47), we have the following off-shell nilpotent ( $s_{ad}^2 = 0$ ) anti-co-BRST (or anti-dual-BRST) symmetry in the theory ( $s_{ad}$ ), namely;

$$\begin{aligned} s_{ad} A_{\mu\nu\eta} &= \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa C^{\rho\sigma}, & s_{ad} C_{\mu\nu} &= -(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu), & s_{ad} \beta_\mu &= -\partial_\mu C_2, \\ s_{ad} C_1 &= -B, & s_{ad} \bar{C}_{\mu\nu} &= \bar{K}_{\mu\nu}, & s_{ad} \bar{\beta}_\mu &= \bar{f}_\mu, & s_{ad} \bar{C}_2 &= -B_2, & s_{ad} \bar{C}_1 &= -B_3, \\ s_{ad} \phi_\mu^{(2)} &= F_\mu, & s_{ad} \bar{F}_\mu &= -\partial_\mu B_3, & s_{ad} f_\mu &= -\partial_\mu B, & s_{ad} K_{\mu\nu} &= \partial_\mu F_\nu - \partial_\nu F_\mu, \\ s_{ad} [\partial^\eta A_{\eta\mu\nu}, \phi_\mu^{(1)}, K_{\mu\nu}, \bar{K}_{\mu\nu}, \bar{K}_{\mu\nu}, B, B_1, B_2, B_3, C_2, \bar{f}_\mu, F_\mu] &= 0, \end{aligned} \quad (49)$$

under which, once again, the total gauge-fixing term remains invariant [i.e.  $s_{ad}(\partial^\mu A_{\mu\nu\eta}) = 0$ ,  $s_{ad} \bar{K}_{\mu\nu} = 0$ ,  $s_{ad} \phi_\mu^{(1)} = 0$ ] and the Lagrangian density  $\mathcal{L}_{(\bar{b})}$  transforms as follows:

$$\begin{aligned} s_{ad} \mathcal{L}_{(\bar{b})} &= \partial_\mu \left[ (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \bar{K}_{\nu\eta} - \bar{K}^{\mu\nu} F_\nu + B_2 (\partial^\mu C_2) - B_3 F^\mu \right. \\ &\quad \left. + B \bar{f}^\mu - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{f}_\nu \right]. \end{aligned} \quad (50)$$

As a consequence of the above, the action integral  $S = \int d^6x \mathcal{L}_{(\bar{b})}$  remains invariant for the well-defined physical fields of the theory that fall off rapidly at infinity.

According to Noether's theorem, the above continuous symmetry invariances lead to the derivation of the following Noether's conserved currents:

$$\begin{aligned} J_{(d)}^\mu &= \frac{1}{2!} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} (\partial_\nu \bar{C}_{\eta\kappa}) K_{\lambda\rho} - (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) K_{\nu\eta} - K^{\mu\nu} \bar{F}_\nu \\ &\quad - B_3 \bar{F}^\mu + B (\partial^\mu \bar{C}_2) + (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) (\partial_\nu \bar{\beta}_\eta - \partial_\eta \bar{\beta}_\nu) \\ &\quad - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) (\partial_\nu \bar{C}_2) + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) f_\nu + B_2 f^\mu, \end{aligned} \quad (51)$$



$$\begin{aligned}
J_{(ad)}^\mu &= -\frac{1}{2!} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} (\partial_\nu C_{\eta\kappa}) \bar{K}_{\lambda\rho} + (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \bar{K}_{\nu\eta} - \bar{K}^{\mu\nu} F_\nu \\
&- B_3 F^\mu + B_2 (\partial^\mu C_2) + (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) (\partial_\nu \beta_\eta - \partial_\eta \beta_\nu) \\
&+ (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) (\partial_\nu C_2) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{f}_\nu + B \bar{f}^\mu.
\end{aligned} \tag{52}$$

The conservation law  $[\partial_\mu J_{(a)d}^\mu = 0]$  can be proven by exploiting the Euler-Lagrange equations of motions (38) and (39) that emerge from the Lagrangian densities  $\mathcal{L}_{(b)}$  and  $\mathcal{L}_{(\bar{b})}$ . The explicit expressions for the conserved charges ( $Q_r = \int d^5x J_{(r)}^0$ ,  $r = d, ad$ ), which are the generators of the (anti-)co-BRST symmetry transformations (49) and (47), are

$$\begin{aligned}
Q_d &= \int d^5x \left[ \frac{1}{2!} \varepsilon^{0ijklm} (\partial_i \bar{C}_{jk}) K_{lm} - (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) \bar{K}_{\nu\eta} - \bar{K}^{0i} \bar{F}_i \right. \\
&+ B_2 \dot{f}^0 + B \dot{\bar{C}}_2 + (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) (\partial_\nu \bar{\beta}_\eta - \partial_\eta \bar{\beta}_\nu) \\
&\left. + (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) f_i - (\partial^0 \beta^i - \partial^i \beta^0) (\partial_i \bar{C}_2) - B_3 \bar{F}^0 \right],
\end{aligned} \tag{53}$$

$$\begin{aligned}
Q_{ad} &= \int d^5x \left[ -\frac{1}{2!} \varepsilon^{0ijklm} (\partial_i C_{jk}) \bar{K}_{lm} + (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) \bar{K}_{\nu\eta} - \bar{K}^{0i} F_i \right. \\
&+ B \dot{\bar{f}}^0 + B_2 \dot{C}_2 + (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) (\partial_\nu \beta_\eta - \partial_\eta \beta_\nu) \\
&\left. - (\partial^0 \beta^i - \partial^i \beta^0) \bar{f}_i + (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i C_2) - B_3 F^0 \right].
\end{aligned} \tag{54}$$

Thus, we conclude that, in addition to the nilpotent (anti-)BRST symmetry transformations, we have another set of nilpotent (anti-)co-BRST symmetry transformations in the theory. However, it is to be emphasized that the (anti-)BRST and (anti-)co-BRST symmetries co-exist together for the Abelian 3-form gauge theory *only* in six  $(5+1)$ -dimensions of spacetime. All these symmetry transformations are fermionic in nature. The distinguishing feature of these symmetries is the invariance of the *total kinetic* and *total gauge-fixing* terms under the (anti-)BRST and (anti-)co-BRST symmetry transformations, respectively. We would like to add that these fermionic symmetry transformations are the most fundamental symmetries in our theory (which would turn out to be a model for the Hodge theory).

We close this section with the following observations. First of all, it can be noted that

$$\begin{aligned}
s_{ad} \mathcal{L}_{(b)} &= \partial_\mu \left[ (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{F}_\nu - C^{\mu\nu} (\partial_\nu B_3) - \bar{C}^{\mu\nu} (\partial_\nu B) - \frac{1}{3} \varepsilon^{\mu\nu\eta\kappa\rho\sigma} F_\nu (\partial_\eta A_{\kappa\rho\sigma}) \right. \\
&+ B \bar{f}^\mu - B_3 F^\mu + \bar{K}^{\mu\nu} f_\nu + B_2 (\partial^\mu C_2) - (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \bar{K}_{\nu\eta} \Big] \\
&+ \left[ \mathcal{K}_{\mu\nu} + \bar{\mathcal{K}}_{\mu\nu} - (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}) \right] (\partial^\mu F^\nu) - (\partial_\mu B) [\bar{f}^\mu + \bar{F}^\mu - \partial^\mu \bar{C}_1] \\
&+ (\partial_\mu B_3) [f^\mu + F^\mu - \partial^\mu C_1] - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \partial_\mu [\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1] \\
&+ \partial_\mu \left[ \mathcal{K}_{\nu\eta} + \bar{\mathcal{K}}_{\nu\eta} - (\partial_\nu \phi_\eta^{(2)} - \partial_\eta \phi_\nu^{(2)}) \right] (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \\
&- \bar{K}^{\mu\nu} \partial_\mu [f_\nu + F_\nu - \partial_\nu C_1],
\end{aligned} \tag{55}$$

$$\begin{aligned}
s_d \mathcal{L}_{(\bar{b})} = & \partial_\mu \left[ -(\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) F_\nu + \bar{C}^{\mu\nu} (\partial_\nu B_3) - C^{\mu\nu} (\partial_\nu B_2) + \frac{1}{3} \varepsilon^{\mu\nu\eta\kappa\rho\sigma} \bar{F}_\nu (\partial_\eta A_{\kappa\rho\sigma}) \right. \\
& + B_2 f^\mu - B_3 \bar{F}^\mu + \mathcal{K}^{\mu\nu} \bar{f}_\nu + B (\partial^\mu \bar{C}_2) + (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) \bar{K}_{\nu\eta} \left. \right] \\
& + \left[ \mathcal{K}_{\mu\nu} + \bar{\mathcal{K}}_{\mu\nu} - (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}) \right] (\partial^\mu \bar{F}^\nu) - (\partial_\mu B_2) [f^\mu + F^\mu - \partial^\mu C_1] \\
& + (\partial_\mu B_3) [\bar{f}^\mu + \bar{F}^\mu - \partial^\mu \bar{C}_1] + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \partial_\mu [f_\nu + F_\nu - \partial_\nu C_1] \\
& - \partial_\mu \left[ \mathcal{K}_{\nu\eta} + \bar{\mathcal{K}}_{\nu\eta} - (\partial_\nu \phi_\eta^{(2)} - \partial_\eta \phi_\nu^{(2)}) \right] (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) \\
& - \mathcal{K}^{\mu\nu} \partial_\mu [\bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1], \tag{56}
\end{aligned}$$

which shows that  $\mathcal{L}_{(b)}$  and  $\mathcal{L}_{(\bar{b})}$  are *equivalent* because both of them respect the off-shell nilpotent (anti-)co-BRST symmetry transformations *together* on the constrained hypersurface [where the field equations (24) and (43) are true]. Furthermore, the off-shell nilpotent (anti-)co-BRST symmetry transformations are absolutely anticommuting in nature *only* on the hypersurface defined by the CF-type field equations. For instance,  $\{s_d, s_{ad}\} A_{\mu\nu\eta} = 0, \{s_d, s_{ad}\} C_{\mu\nu} = 0, \{s_d, s_{ad}\} \bar{C}_{\mu\nu} = 0$  only when CF-type restrictions (24) and (43) are taken into account. Thus, we conclude that the hallmark of a  $p$ -form ( $p = 1, 2, 3, \dots$ ) gauge theory, within the framework of BRST formalism, is the existence of CF-type restrictions. In fact, in our earlier works [16,22], we have provided mathematical basis for the existence of CF-type restrictions and their connection with the concept of geometrical object called gerbes. Finally, it can be checked that (anti-)co-BRST symmetry transformations (49) and (47) can be derived from the analogue of (42) where  $Q_{(a)b}$  should be replaced by  $Q_{(a)d}$ . Thus, conserved (anti-)co-BRST charges  $Q_{(a)d}$  [of (54) and (53)] are the generator of transformations (49) and (47).

## 5 Bosonic symmetry: conserved charge

As has been pointed out earlier, there are four fermionic (i.e. nilpotent) symmetry transformations  $(s_{(a)b}, s_{(a)d})$  in the theory. The following anticommutators, between two of the above fermionic symmetry transformations, define a new bosonic symmetry in the theory. These anticommutators (between suitable fermionic symmetries) are as follows:

$$\{s_b, s_d\} = s_\omega, \quad \{s_{ab}, s_{ad}\} = s_{\bar{\omega}}. \tag{57}$$

Rest of the anticommutators of the fermionic symmetry transformations do not define any new symmetry. For instance, we have already seen that  $\{s_b, s_{ab}\} = 0, \{s_d, s_{ad}\} = 0$  on the constrained hypersurface (in the 6D spacetime manifold) described by the CF-type field equations. In exactly similar fashion, as it turns out, the anticommutators  $\{s_b, s_{ad}\} = 0$  and  $\{s_d, s_{ab}\} = 0$  are also zero modulo a  $U(1)$  vector gauge transformations (see, e.g., [30] for details). Thus, for all practical purposes, the absolute anticommutativity amongst the fermionic symmetry transformations  $s_{(a)b}$  and  $s_{(a)d}$  is true except the anticommutators in (57). The latter anticommutators define a single bosonic symmetry [cf. (64) below].

The bosonic symmetry transformations  $s_\omega$  yield the following infinitesimal transforma-

tions for the relevant fields of the theory, namely;

$$\begin{aligned}
s_\omega A_{\mu\nu\eta} &= \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} (\partial^\kappa K^{\rho\sigma}) + \left( \partial_\mu \mathcal{K}_{\nu\eta} + \partial_\nu \mathcal{K}_{\eta\mu} + \partial_\eta \mathcal{K}_{\mu\nu} \right), & s_\omega \beta_\mu &= \partial_\mu B, \\
s_\omega C_{\mu\nu} &= -(\partial_\mu f_\nu - \partial_\nu f_\mu), & s_\omega \bar{C}_{\mu\nu} &= \partial_\mu \bar{F}_\nu - \partial_\nu \bar{F}_\mu, & s_\omega \bar{\beta}_\mu &= \partial_\mu B_2, \\
s_\omega [B, B_1, B_2, B_3, C_1, \bar{C}_1, C_2, \bar{C}_2, \phi_\mu^{(1)}, \phi_\mu^{(2)}, f_\mu, \bar{f}_\mu, F_\mu, \bar{F}_\mu, K_{\mu\nu}, \bar{K}_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}] &= 0. \quad (58)
\end{aligned}$$

The above transformations are symmetry transformations because the Lagrangian density  $\mathcal{L}_{(b)}$  remains quasi-invariant as it transforms to a total spacetime derivative:

$$\begin{aligned}
s_\omega \mathcal{L}_{(b)} &= \partial_\mu \left[ (\partial^\mu \mathcal{K}^{\nu\eta} + \partial^\nu \mathcal{K}^{\eta\mu} + \partial^\eta \mathcal{K}^{\mu\nu}) K_{\nu\eta} - (\partial^\mu K^{\nu\eta} + \partial^\nu K^{\eta\mu} + \partial^\eta K^{\mu\nu}) \mathcal{K}_{\nu\eta} \right. \\
&\quad \left. - B_2 (\partial^\mu B) + B (\partial^\mu B_2) + (\partial^\mu f^\nu - \partial^\nu f^\mu) \bar{F}_\nu + (\partial^\mu \bar{F}^\nu - \partial^\nu \bar{F}^\mu) f_\nu \right]. \quad (59)
\end{aligned}$$

Similarly, under the infinitesimal bosonic symmetry transformations ( $s_{\bar{\omega}}$ ):

$$\begin{aligned}
s_{\bar{\omega}} A_{\mu\nu\eta} &= \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} (\partial^\kappa \bar{K}^{\rho\sigma}) + \left( \partial_\mu \bar{\mathcal{K}}_{\nu\eta} + \partial_\nu \bar{\mathcal{K}}_{\eta\mu} + \partial_\eta \bar{\mathcal{K}}_{\mu\nu} \right), & s_{\bar{\omega}} \beta_\mu &= -\partial_\mu B, \\
s_{\bar{\omega}} C_{\mu\nu} &= -(\partial_\mu F_\nu - \partial_\nu F_\mu), & s_{\bar{\omega}} \bar{C}_{\mu\nu} &= \partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu, & s_{\bar{\omega}} \bar{\beta}_\mu &= -\partial_\mu B_2, \\
s_{\bar{\omega}} [B, B_1, B_2, B_3, C_1, \bar{C}_1, C_2, \bar{C}_2, \phi_\mu^{(1)}, \phi_\mu^{(2)}, f_\mu, \bar{f}_\mu, F_\mu, \bar{F}_\mu, K_{\mu\nu}, \bar{K}_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}] &= 0, \quad (60)
\end{aligned}$$

the Lagrangian density  $\mathcal{L}_{(\bar{b})}$  transforms to a total spacetime derivative as follows:

$$\begin{aligned}
s_{\bar{\omega}} \mathcal{L}_{(\bar{b})} &= -\partial_\mu \left[ (\partial^\mu \bar{\mathcal{K}}^{\nu\eta} + \partial^\nu \bar{\mathcal{K}}^{\eta\mu} + \partial^\eta \bar{\mathcal{K}}^{\mu\nu}) \bar{K}_{\nu\eta} - (\partial^\mu \bar{K}^{\nu\eta} + \partial^\nu \bar{K}^{\eta\mu} + \partial^\eta \bar{K}^{\mu\nu}) \bar{\mathcal{K}}_{\nu\eta} \right. \\
&\quad \left. - B_2 (\partial^\mu B) + B (\partial^\mu B_2) + (\partial^\mu \bar{f}^\nu - \partial^\nu \bar{f}^\mu) F_\nu + (\partial^\mu F^\nu - \partial^\nu F^\mu) \bar{f}_\nu \right]. \quad (61)
\end{aligned}$$

We note [from (59) and (61)] that the equivalent action integrals ( $S_1 = \int d^6x \mathcal{L}_{(b)}$ ,  $S_2 = \int d^6x \mathcal{L}_{(\bar{b})}$ ) remain invariant under the transformations  $s_\omega$  and  $s_{\bar{\omega}}$  for the physical fields of the theory which fall off rapidly at infinity (due to Gauss's divergence theorem).

It is to be pointed out that the following transformations are also true, namely;

$$\begin{aligned}
s_\omega \mathcal{L}_{(\bar{b})} &= \partial_\mu \left[ (\partial^\mu K^{\nu\eta} + \partial^\nu K^{\eta\mu} + \partial^\eta K^{\mu\nu}) \bar{\mathcal{K}}_{\nu\eta} - (\partial^\mu \mathcal{K}^{\nu\eta} + \partial^\nu \mathcal{K}^{\eta\mu} + \partial^\eta \mathcal{K}^{\mu\nu}) \bar{K}_{\nu\eta} \right. \\
&\quad \left. - (\partial^\mu f^\nu - \partial^\nu f^\mu) \bar{f}_\nu - (\partial^\mu \bar{F}^\nu - \partial^\nu \bar{F}^\mu) F_\nu + B (\partial^\mu B_2) - B_2 (\partial^\mu B) \right] \\
&\quad + \partial_\mu \left[ K_{\nu\eta} + \bar{K}_{\nu\eta} - (\partial_\nu \phi_\eta^{(1)} - \partial_\eta \phi_\nu^{(1)}) \right] (\partial^\mu \mathcal{K}^{\nu\eta} + \partial^\nu \mathcal{K}^{\eta\mu} + \partial^\eta \mathcal{K}^{\mu\nu}) \\
&\quad - \partial_\mu \left[ \bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1 \right] (\partial^\mu f^\nu - \partial^\nu f^\mu), \quad (62)
\end{aligned}$$

$$\begin{aligned}
s_{\bar{\omega}} \mathcal{L}_{(b)} &= \partial_\mu \left[ (\partial^\mu \bar{\mathcal{K}}^{\nu\eta} + \partial^\nu \bar{\mathcal{K}}^{\eta\mu} + \partial^\eta \bar{\mathcal{K}}^{\mu\nu}) K_{\nu\eta} - (\partial^\mu \bar{K}^{\nu\eta} + \partial^\nu \bar{K}^{\eta\mu} + \partial^\eta \bar{K}^{\mu\nu}) \mathcal{K}_{\nu\eta} \right. \\
&\quad \left. + (\partial^\mu \bar{f}^\nu - \partial^\nu \bar{f}^\mu) f_\nu + (\partial^\mu F^\nu - \partial^\nu F^\mu) \bar{F}_\nu - B (\partial^\mu B_2) + B_2 (\partial^\mu B) \right] \\
&\quad - \partial_\mu \left[ \mathcal{K}_{\nu\eta} + \bar{\mathcal{K}}_{\nu\eta} - (\partial_\nu \phi_\eta^{(2)} - \partial_\eta \phi_\nu^{(2)}) \right] (\partial^\mu K^{\nu\eta} + \partial^\nu K^{\eta\mu} + \partial^\eta K^{\mu\nu}) \\
&\quad - \partial_\mu \left[ \bar{f}_\nu + \bar{F}_\nu - \partial_\nu \bar{C}_1 \right] (\partial^\mu f^\nu - \partial^\nu f^\mu). \quad (63)
\end{aligned}$$

Thus, both the coupled Lagrangian densities  $\mathcal{L}_{(b)}$  and  $\mathcal{L}_{(\bar{b})}$  respect the bosonic symmetries  $s_\omega$  and  $s_{\bar{\omega}}$  on the constrained hypersurface defined by the CF-type field equations (24) and (43). Mention can also be made of the key observation that, even though,  $s_\omega$  and  $s_{\bar{\omega}}$  look quite different [cf. (58), (60)], they are actually equivalent on the constrained hypersurface (defined in the 6D Minkowskian spacetime manifold) where the CF-type restrictions (24) and (43) are true. In fact, as it turns out, we have

$$s_\omega + s_{\bar{\omega}} = 0 \implies s_\omega = -s_{\bar{\omega}}, \quad (64)$$

which shows that *only one* of  $s_\omega$  and  $s_{\bar{\omega}}$  is an independent bosonic symmetry transformation in the theory. Thus, henceforth, we shall take  $s_\omega$  as the *only* bosonic symmetry.

According to Noether's theorem, the above local, infinitesimal and continuous symmetry transformation leads to the derivation of the following Noether's conserved current (w.r.t. the Lagrangian density  $\mathcal{L}_{(b)}$ ), namely;

$$\begin{aligned} J_{(\omega)}^\mu &= \frac{1}{2!} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} (\partial_\nu K_{\eta\kappa}) K_{\lambda\rho} - \frac{1}{2!} \varepsilon^{\mu\nu\eta\kappa\lambda\rho} (\partial_\nu \mathcal{K}_{\eta\kappa}) \mathcal{K}_{\lambda\rho} \\ &- (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) (\partial_\nu f_\eta - \partial_\eta f_\nu) - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) (\partial_\nu B) \\ &- (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) (\partial_\nu \bar{F}_\eta - \partial_\eta \bar{F}_\nu) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) (\partial_\nu B_2). \end{aligned} \quad (65)$$

The conservation law ( $\partial_\mu J_{(\omega)}^\mu = 0$ ) can be proven by exploiting the equations of motion (38) derived from Lagrangian density  $\mathcal{L}_{(b)}$ . Exploiting the usual tricks of quantum field theory, it can be checked that the exact expression for the conserved charge is

$$\begin{aligned} Q_\omega &= \int d^5x \left[ \frac{1}{2!} \varepsilon^{0ijklm} (\partial_i K_{jk}) K_{lm} - \frac{1}{2!} \varepsilon^{0ijklm} (\partial_i \mathcal{K}_{jk}) \mathcal{K}_{lm} \right. \\ &- (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) (\partial_\nu f_\eta - \partial_\eta f_\nu) - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i B) \\ &- \left. (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) (\partial_\nu \bar{F}_\eta - \partial_\eta \bar{F}_\nu) - (\partial^0 \beta^i - \partial^i \beta^0) (\partial_i B_2) \right]. \end{aligned} \quad (66)$$

The interesting point to be stressed is the origin of the derivation of conserved charge  $Q_\omega$  that has emerged out from the bosonic symmetry transformations corresponding to  $s_\omega$ . The latter is equal to the anticommutator of two nilpotent (fermionic) symmetry transformations. As pointed out earlier, the above nilpotent symmetry transformations are the analogue of the (co-)exterior derivatives of differential geometry. As a consequence, the anticommutator of the suitable nilpotent symmetry transformations (that is equivalent to a bosonic symmetry transformation) is the analogue of the Laplacian operator of differential geometry. We shall see, in Sec. VII, that the conserved charge  $Q_\omega$  provides an accurate physical realization of the Laplacian operator of differential geometry from the point of view of the ghost number consideration as well as the specific algebra obeyed by it. Finally, it can be verified that, besides being the Casimir operator for the whole algebra,  $Q_\omega$  is also the generator of transformations (58) if we exploit the analogue of (42) appropriately.

Before we wrap up this section, we would like to state that, for the present higher dimensional (i.e.  $D = 6$ ) and higher-form (i.e.  $p = 3$ ) Abelian gauge theory, we have:

$$\begin{aligned} s_\omega^{(1)} &= \{s_b, s_{ad}\} \bar{\beta}_\mu = \partial_\mu (B_1 - B_3), & s_\omega^{(2)} &= \{s_{ab}, s_d\} \beta_\mu = \partial_\mu (B_1 + B_3), \\ s_\omega^{(1)} &= \{s_b, s_{ad}\} \phi_\mu^{(1)} = -\partial_\mu B, & s_\omega^{(2)} &= \{s_{ab}, s_d\} \phi_\mu^{(1)} = \partial_\mu B_2, \\ s_\omega^{(1)} &= \{s_b, s_{ad}\} \phi_\mu^{(2)} = -\partial_\mu B, & s_\omega^{(2)} &= \{s_{ab}, s_d\} \phi_\mu^{(2)} = -\partial_\mu B_2. \end{aligned} \quad (67)$$

For the rest of the fields of the theory, it can be checked that  $\{s_b, s_{ad}\} = 0$ ,  $\{s_d, s_{ab}\} = 0$ . As mentioned earlier, we already know that  $\{s_b, s_{ab}\} = 0$ ,  $\{s_d, s_{ad}\} = 0$  on the hypersurface defined by the CF-type field equations (24) and (43). Thus, we conclude that, out of all the possible anticommutators between  $s_{(a)b}$  and  $s_{(a)d}$ , only two of them define a bosonic symmetry (i.e.  $s_\omega = \{s_b, s_d\} = -\{s_{ab}, s_{ad}\}$ ) and rest of them have absolute anticommutativity property only up to a  $U(1)$  vector gauge transformations. We would like to lay emphasis on the fact that this observation, in the context of 6D Abelian 3-form gauge theory, is totally *different* from our experience in 2D Abelian 1-form and 4D Abelian 2-form gauge theories (see, e.g. [10-16]) where there exists an absolute anticommutativity between the nilpotent transformations  $s_{(a)b}$  and  $s_{(a)d}$  except  $s_\omega = \{s_b, s_d\} = -\{s_{ab}, s_{ad}\}$  which defines the bosonic symmetry. Furthermore, it can be checked explicitly that the transformations  $\{s_b, s_{ad}\}$  and  $\{s_d, s_{ab}\}$  [cf. (67)] are, even though, the symmetry transformations for the Lagrangian densities  $\mathcal{L}_{(b)}$  and/or  $\mathcal{L}_{(\bar{b})}$ , these *do not* commute with  $s_g$  (see, e.g., (72) below). We discuss about these symmetries and connected issues in our Appendix E. Our *final remark* is the fact that one can choose  $B = B_1 = B_2 = B_3 = 0$  so that  $s_\omega^{(1)} = s_\omega^{(2)} = 0$  in equation (67). This choice, it may be emphasized, does not spoil the existence of the fundamental fermionic (i.e. nilpotent) symmetries  $s_{(a)b}$  and  $s_{(a)d}$  which are the analogues of the nilpotent (co-)exterior derivatives of differential geometry.

## 6 Discrete and ghost-scale symmetries

A close look at the Lagrangian densities (30) and (31) demonstrates that the non-ghost part of  $\mathcal{L}_{(b)}$  and  $\mathcal{L}_{(\bar{b})}$  remains invariant under the following discrete symmetry transformations:

$$\begin{aligned} A_{\mu\nu\eta} &\rightarrow \pm \frac{i}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}, & K_{\mu\nu} &\rightarrow \pm i \mathcal{K}_{\mu\nu}, & \mathcal{K}_{\mu\nu} &\rightarrow \pm i K_{\mu\nu}, & \bar{K}_{\mu\nu} &\rightarrow \pm i \bar{\mathcal{K}}_{\mu\nu}, \\ \bar{\mathcal{K}}_{\mu\nu} &\rightarrow \pm i \bar{K}_{\mu\nu}, & \phi_\mu^{(1)} &\rightarrow \pm i \phi_\mu^{(2)}, & \phi_\mu^{(2)} &\rightarrow \pm i \phi_\mu^{(1)}, & B_1 &\rightarrow \pm i B_3, & B_3 &\rightarrow \pm i B_1. \end{aligned} \quad (68)$$

In exactly similar fashion, the ghost part of the Lagrangian densities (30) and (31) respect the following discrete symmetry transformations:

$$\begin{aligned} C_{\mu\nu} &\rightarrow \pm i \bar{C}_{\mu\nu}, & \bar{C}_{\mu\nu} &\rightarrow \pm i C_{\mu\nu}, & \beta_\mu &\rightarrow \pm i \bar{\beta}_\mu, & \bar{\beta}_\mu &\rightarrow \mp i \beta_\mu, & C_2 &\rightarrow \pm i \bar{C}_2, \\ \bar{C}_2 &\rightarrow \pm i C_2, & C_1 &\rightarrow \pm i \bar{C}_1, & \bar{C}_1 &\rightarrow \pm i C_1, & B &\rightarrow \mp i B_2, & B_2 &\rightarrow \pm i B, \\ f_\mu &\rightarrow \pm i \bar{F}_\mu, & \bar{F}_\mu &\rightarrow \pm i f_\mu, & \bar{f}_\mu &\rightarrow \pm i F_\mu, & F_\mu &\rightarrow \pm i \bar{f}_\mu. \end{aligned} \quad (69)$$

It is obvious now that the total Lagrangian densities (30) and (31) remain invariant under the combined discrete symmetry transformations listed in (68) and (69). We shall see, in the next section, that the above discrete symmetry transformations play a key role in providing the physical realization of Hodge duality (\*) operation of differential geometry.

It is to be noted that the decisive feature of the discrete symmetry transformations, in the gauge sector [cf. (68)] of the Lagrangian densities, has been the *self-duality* condition on the gauge field [cf. (14)] which is intimately connected with the Hodge duality (\*) operation. This is the reason that the generalization of  $A_{\mu\nu\eta} \rightarrow -\frac{1}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma}$  in the gauge sector (together with the discrete symmetry transformations in the ghost sector) provides the physical realization of the Hodge duality (\*) operation of differential geometry. In fact, the

discrete symmetry transformations (68) and (69) are intimately connected with the original discrete symmetry transformation in (14) (which is nothing but the self-duality condition).

The ghost part of the Lagrangian densities (30) and (31), in addition to the discrete transformations (69), respect a continuous scale symmetry transformations as listed below

$$\begin{aligned} C_{\mu\nu} &\rightarrow e^{+\Lambda} C_{\mu\nu}, & \bar{C}_{\mu\nu} &\rightarrow e^{-\Lambda} \bar{C}_{\mu\nu}, & \beta_\mu &\rightarrow e^{+2\Lambda} \beta_\mu, & \bar{\beta}_\mu &\rightarrow e^{-2\Lambda} \bar{\beta}_\mu, & C_2 &\rightarrow e^{+3\Lambda} C_2, \\ \bar{C}_2 &\rightarrow e^{-3\Lambda} \bar{C}_2, & C_1 &\rightarrow e^{+\Lambda} C_1, & \bar{C}_1 &\rightarrow e^{-\Lambda} \bar{C}_1, & B &\rightarrow e^{+2\Lambda} B, & B_2 &\rightarrow e^{-2\Lambda} B_2, \\ f_\mu &\rightarrow e^{+\Lambda} f_\mu, & \bar{f}_\mu &\rightarrow e^{-\Lambda} \bar{f}_\mu, & F_\mu &\rightarrow e^{+\Lambda} F_\mu, & \bar{F}_\mu &\rightarrow e^{-\Lambda} \bar{F}_\mu, \end{aligned} \quad (70)$$

where  $\Lambda$  is a global (spacetime independent) parameter and numerals ( $\mp 1, \mp 2, \mp 3$ ) in the exponentials stand for the ghost number of the (anti-)ghost fields. According to the basic tenets of BRST formalism, all the rest of the fields (in the gauge sector of the Lagrangian densities) are endowed with the ghost number equal to zero. As a consequence, the ghost-scale symmetry transformations, on the generic field  $\Psi$  of this sector, is:

$$\Psi \longrightarrow \Psi' = e^{0 \cdot \Lambda} \Psi \implies \Psi' = \Psi, \quad (71)$$

where the generic field  $\Psi$  stands for  $\Psi = A_{\mu\nu\eta}, K_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}, B_1, B_3, \phi_\mu^{(1)}, \phi_\mu^{(2)}$ . Choosing  $\Lambda = 1$  in the scale symmetry transformations (70) and (71), we obtain the following infinitesimal ghost-scale symmetry transformations:

$$\begin{aligned} s_g C_{\mu\nu} &= +C_{\mu\nu}, & s_g \bar{C}_{\mu\nu} &= -\bar{C}_{\mu\nu}, & s_g \beta_\mu &= +2\beta_\mu, & s_g \bar{\beta}_\mu &= -2\bar{\beta}_\mu, \\ s_g C_2 &= +3C_2, & s_g \bar{C}_2 &= -3\bar{C}_2, & s_g C_1 &= +C_1, & s_g \bar{C}_1 &= -\bar{C}_1, & s_g B &= +2B, \\ s_g B_2 &= -2B_2, & s_g f_\mu &= +f_\mu, & s_g \bar{f}_\mu &= -\bar{f}_\mu, & s_g F_\mu &= +F_\mu, & s_g \bar{F}_\mu &= -\bar{F}_\mu, \\ s_g [A_{\mu\nu\eta}, K_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}, \phi_\mu^{(1)}, \phi_\mu^{(2)}, B_1, B_3] &= 0, \end{aligned} \quad (72)$$

where  $s_g$  is the infinitesimal version of the ghost-scale transformations (70) and (71).

According to Noether's theorem, the continuous symmetry transformations (72) lead to the derivation of the Noether ghost conserved current as given below

$$\begin{aligned} J_{(g)}^\mu &= (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu}) C_{\nu\eta} + (\partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu}) \bar{C}_{\nu\eta} \\ &- 2(\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \beta_\nu + 2(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{\beta}_\nu - 2B_2 \beta^\mu - \bar{C}^{\mu\nu} f_\nu \\ &- 2B \bar{\beta}^\mu - C_1 \bar{F}^\mu - \bar{C}_1 f^\mu + 3C_2 (\partial^\mu \bar{C}_2) + 3\bar{C}_2 (\partial^\mu C_2) - C^{\mu\nu} \bar{F}_\nu. \end{aligned} \quad (73)$$

The conservation law  $\partial_\mu J_{(g)}^\mu = 0$  can be proven by exploiting the equations of motion (38) and (39) for the (anti-)ghost fields of our present theory. The conserved current  $J_{(g)}^\mu$  leads to the derivation of the conserved ghost charge ( $Q_g = \int d^5x J_{(g)}^0$ ) as:

$$\begin{aligned} Q_g &= \int d^5x \left[ (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) C_{\nu\eta} + (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) \bar{C}_{\nu\eta} \right. \\ &- 2(\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \beta_i + 2(\partial^0 \beta^i - \partial^i \beta^0) \bar{\beta}_i - 2B_2 \beta^0 - C^{0i} \bar{F}_i - 2B \bar{\beta}^0 - C_1 \bar{F}^0 \\ &- \bar{C}_1 f^0 + 3C_2 \dot{\bar{C}}_2 + 3\bar{C}_2 \dot{C}_2 - \bar{C}^{0i} f_i \left. \right]. \end{aligned} \quad (74)$$

The charge  $Q_g$  is the generator of infinitesimal transformations (72) if we exploit the power and potential of the analogue of relationship (42) by using the canonical brackets, that are derived from the Lagrangian density  $\mathcal{L}_{(b)}$ , in the evaluation of the commutators.

## 7 Cohomological aspects: algebraic structures

We have noted, thus far, that there exist *six* continuous symmetries in the theory. Four of them (i.e.  $s_{(a)b}$ ,  $s_{(a)d}$ ) are fermionic (nilpotent) in nature and two of them ( $s_\omega$ ,  $s_g$ ) are bosonic in nature. We can verify, in a straightforward manner, that the operator form of these continuous symmetry transformations satisfy the following structures, namely;

$$\begin{aligned} s_{(a)b}^2 &= 0, & s_{(a)d}^2 &= 0, & \{s_b, s_{ab}\} &= 0, & \{s_d, s_{ad}\} &= 0, & \{s_b, s_{ad}\} &= 0, \\ [s_g, s_b] &= +s_b, & [s_g, s_{ab}] &= -s_{ab}, & [s_g, s_d] &= -s_d, & [s_g, s_{ad}] &= +s_{ad}, \\ \{s_{ab}, s_d\} &= 0, & \{s_b, s_d\} &= s_\omega = -\{s_{ab}, s_{ad}\}, & [s_\omega, s_r] &= 0 & r &= g, b, ab, d, ad, \end{aligned} \quad (75)$$

where we have taken the infinitesimal versions of the continuous symmetry transformations (32), (34), (49), (47), (58) and (72). We also note that we mean by  $s_b^2 = 0$  (in the operator form) as  $\frac{1}{2}\{s_b, s_b\}\Psi \equiv \frac{1}{2}(s_b s_b + s_b s_b)\Psi = 0$ ,  $\{s_b, s_{ab}\}\Psi \equiv (s_b s_{ab} + s_{ab} s_b)\Psi = 0$ ,  $[s_\omega, s_r]\Psi \equiv (s_\omega s_r - s_r s_\omega)\Psi = 0$ , etc., where  $\Psi$  is the generic field of the theory.

A close look at the algebra (75) establishes the fact that this algebra is the analogue of the algebra satisfied by the de Rham cohomological operators  $(d, \delta, \Delta)$  of differential geometry as one knows that the standard algebra, obeyed by the exterior derivative  $d$ , co-exterior derivative  $\delta$  and the Laplacian operator  $\Delta = (d + \delta)^2 = (d\delta + \delta d)$ , is [17-20]:

$$d^2 = \delta^2 = 0, \quad \{d, \delta\} = \Delta = (d + \delta)^2, \quad [\Delta, d] = 0, \quad [\Delta, \delta] = 0. \quad (76)$$

An accurate comparison of equations (75) and (76) establishes the fact the set  $(s_b, s_d, s_\omega)$  and  $(s_{ab}, s_{ad}, -s_\omega)$  are the analogues of the de Rham cohomological operators  $(d, \delta, \Delta)$  of differential geometry where there exists a two-to-one mapping because  $(s_b, s_{ad}) \longrightarrow d$ ,  $(s_d, s_{ab}) \longrightarrow \delta$  and  $\{s_b, s_d\} = -\{s_{ab}, s_{ad}\} = s_\omega \implies \Delta$ , at the algebraic level.

Even though, there is a perfect matching between  $(d, \delta, \Delta)$  and the transformations  $s_{(a)d}$ ,  $s_{(a)b}$  and  $s_\omega$  at the algebraic level, there are a couple of points which are missing as far as the perfect analogy is concerned. First, as we know, the co-exterior derivative  $\delta$  is connected to the exterior derivative  $d$  by the relation:  $\delta = \pm * d *$  where  $(*)$  is the Hodge duality operation. Thus, we have to provide the physical realization of  $(*)$  in the language of symmetry properties. Second, we know that the (co-)exterior derivatives (i.e.  $(\delta)d$ ) (lower)raise the degree of a form by one when they operate on the latter whereas the action of the Laplacian operator does not change the degree of a form at all. We should be able to provide the analogy of the above observations in the language of symmetry properties if we have to prove that the 6D Abelian 3-form gauge theory is a perfect field theoretic model for the Hodge theory where there exist appropriate physical realizations of the cohomological operators  $(d, \delta, \Delta)$  in terms of the symmetry properties of the theory.

We address the first of the above issues in the following manner. As stated earlier, the combination of the discrete symmetry transformations (68) and (69) provide the analogue of the Hodge duality  $(*)$  operation because the following relationships are true:

$$s_d \Psi = \pm * s_b * \Psi, \quad s_{ad} \Psi = \pm * s_{ab} * \Psi, \quad (77)$$

where  $\Psi$  is the generic field of the theory and  $*$  is the combined discrete symmetry transformations. Thus, we note that the interplay of the continuous and discrete symmetry

transformations of the theory provide the analogue of connection between the co-exterior derivative ( $\delta$ ) and exterior derivative ( $d$ ) of differential geometry (i.e.  $\delta = \pm * d *$ ). The  $(\pm)$  signs in relation  $\delta = \pm * d *$  are determined by the dimensions of the spacetime manifold and the degree of the forms that are involved in an inner product (in the realm of differential geometry [17-20]). We have to provide the physical origin for  $(\pm)$  signs.

The  $(\pm)$  signs of equation (77) are decided by a couple of successive operations of the discrete symmetry transformations (68) and (69) on the generic field [i.e.  $*(\Psi) = \pm \Psi$ ] (see, e.g., [20]). As it turns out, only four fields  $\beta_\mu, \bar{\beta}_\mu, B, B_2$  are the ones that possess  $(+)$  sign after two successive operations of discrete symmetry transformations. The rest of the fields carry  $(-)$  sign after the above successive operations. Furthermore, the CF-type of restrictions of (24) and (43) remain invariant under the combined discrete symmetry operations (68) and (69). This shows the fact that our present theory is a *self-dual* theory where the CF-type restrictions (which are responsible for the absolute anticommutativity and the existence of the coupled Lagrangian densities) are physical and duality-invariant.

We address now the second of the issues that have been raised above. The analogue of the degree of a form (in differential geometry) is the ghost number defined as:  $i Q_g |\chi\rangle_n = n |\chi\rangle_n$  where  $|\chi\rangle_n$  is a non-trivial state with ghost number equal to  $n$  (in the total quantum Hilbert space of states). It can be checked that the following relations are true:

$$\begin{aligned} i Q_g Q_b |\chi\rangle_n &= (n+1) Q_b |\chi\rangle_n, & i Q_g Q_{ad} |\chi\rangle_n &= (n+1) Q_{ad} |\chi\rangle_n, \\ i Q_g Q_d |\chi\rangle_n &= (n-1) Q_d |\chi\rangle_n, & i Q_g Q_{ab} |\chi\rangle_n &= (n-1) Q_{ab} |\chi\rangle_n, \\ i Q_g Q_\omega |\chi\rangle_n &= n Q_\omega |\chi\rangle_n, \end{aligned} \quad (78)$$

where the conserved charges  $Q_{(a)b}, Q_{(a)d}, Q_\omega$  and  $Q_g$  satisfy the analogue of algebraic relations, that are satisfied by the operator form of the symmetry transformations (75), as

$$\begin{aligned} Q_{(a)b}^2 &= 0, & Q_{(a)d}^2 &= 0, & \{Q_b, Q_d\} &= Q_\omega = -\{Q_{ad}, Q_{ab}\}, \\ [Q_\omega, Q_r] &= 0 \quad (r = b, ab, d, ad, g, \omega), & \{Q_b, Q_{ab}\} &= 0, \\ \{Q_d, Q_{ad}\} &= 0, & \{Q_b, Q_{ab}\} &= 0, & \{Q_d, Q_{ab}\} &= 0, \\ i [Q_g, Q_b] &= + Q_b, & i [Q_g, Q_{ab}] &= - Q_{ab}, \\ i [Q_g, Q_{ad}] &= + Q_{ad}, & i [Q_g, Q_d] &= - Q_d. \end{aligned} \quad (79)$$

Thus, we note that the ghost numbers of  $Q_b |\chi\rangle_n, Q_d |\chi\rangle_n$  and  $Q_\omega |\chi\rangle_n$  are  $(n+1), (n-1)$  and  $n$ , respectively. Similarly, the states  $Q_{ad} |\chi\rangle_n, Q_{ab} |\chi\rangle_n$  and  $Q_\omega |\chi\rangle_n$  are endowed with ghost numbers equal to  $(n+1), (n-1)$  and  $n$ , respectively. In our Appendices B and C, we have given simple proofs of the validity of  $Q_{(a)b}^2 = 0, \{Q_b, Q_{ab}\} = 0$ , etc., by using the concept of conserved charges as the generators for the continuous nilpotent symmetry transformations. This way, one can also prove the rest of the relations of (79). Finally, we point out that, if the ghost number of a state is identified with the degree of a form, then, we have two-to-one mappings:  $(Q_b, Q_{ad}) \rightarrow d, (Q_{ab}, Q_d) \rightarrow \delta$  and  $Q_\omega = \{Q_b, Q_d\} = -\{Q_{ab}, Q_{ad}\} \rightarrow \Delta$  between the conserved charges and the cohomological operators of differential geometry.

There is yet another link that we have *not* been able to establish within the framework of BRST formalism in the language of symmetry properties and their corresponding conserved charges. This issue is connected with the Hodge decomposition theorem which states that, on a manifold without a boundary, one can uniquely decompose a given  $n$ -form ( $f_n$ ) as a



sum of an exact form (i.e.  $de_{n-1}$ ), a co-exact form ( $\delta c_{n+1}$ ) and a harmonic form  $w_n$  (i.e.  $dw_n = 0$ ,  $\delta w_n = 0 \Rightarrow \Delta w_n = 0$ ) as [17-20]

$$f_n = w_n + d e_{n-1} + \delta c_{n+1}, \quad (80)$$

where the de Rham cohomological operators ( $d, \delta, \Delta$ ) of differential geometry play a very decisive role in the above celebrated Hodge decomposition theorem.

The above issue can also be addressed within the framework of BRST formalism in the space of quantum Hilbert space of states. In fact, any arbitrary state  $|\xi\rangle_n$  with a ghost number  $n$  (i.e.  $i Q_g |\xi\rangle_n = n |\xi\rangle_n$ ) can be decomposed uniquely, in the quantum Hilbert space of states, in terms of a BRST exact state  $Q_b |\zeta\rangle_{n-1}$ , a co-exact state  $Q_d |\kappa\rangle_{n+1}$  and a harmonic state  $|w\rangle_n$  ( $Q_w |w\rangle_n = 0 \Rightarrow Q_b |w\rangle_n = 0$ ,  $Q_d |w\rangle_n = 0$ ) as given below

$$|\xi\rangle_n = |w\rangle_n + Q_b |\zeta\rangle_{n-1} + Q_d |\kappa\rangle_{n+1}. \quad (81)$$

Due to two-to-one mapping between the conserved charges and the cohomological operators:  $(Q_b, Q_{ad}) \rightarrow d$ ,  $(Q_d, Q_{ab}) \rightarrow \delta$ ,  $(Q_w, -Q_w) \rightarrow \Delta$ , it is straightforward to re-express (81) in an alternative way (in the total quantum Hilbert space of states) as illustrated below:

$$|\xi\rangle_n = |w\rangle_n + Q_{ad} |\zeta\rangle_{n-1} + Q_{ab} |\kappa\rangle_{n+1}, \quad (82)$$

where  $|w\rangle_n$  is the harmonic state (i.e. the most *symmetric* state in the whole theory) as it is (anti-)BRST as well as (anti-)co-BRST invariant. In other words, it obeys:

$$Q_{a(b)} |w\rangle_n = 0, \quad Q_{a(d)} |w\rangle_n = 0. \quad (83)$$

Thus, we note that  $|w\rangle_n$  (i.e. the harmonic state) can be chosen to be the physical state  $|phys\rangle$  of the theory which respects all the *four* basic symmetries ( $s_{(a)b}, s_{(a)d}$ ).

Now we dwell a bit on the constraints of the theory. First of all, we know that the original Lagrangian density ( $\mathcal{L}_0 = \frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa}$ ) is a singular Lagrangian density that supports a set of primary and secondary constraints which are of first-class variety in the language of Dirac's prescription for the classification scheme [31,32]. These are  $\Pi^{0\mu\nu} = (1/3) H^{00\mu\nu} \approx 0$ , and  $\partial_i H^{0ijk} \approx 0$  which finally imply  $\Pi^{ij} \equiv (1/3) H^{00ij} \approx 0$ ,  $\Pi^{0i} = (1/3) H^{000i} \approx 0$  and  $\dot{\Pi}^{ij} = -\partial_k H^{0ijk} \approx 0$ . Taking the physical state as the harmonic state (i.e.  $|w\rangle = |phys\rangle$ ), it can be seen that the following operator form of the first-class constraints of the original Lagrangian density annihilate the physical state  $|phys\rangle$ , namely;

$$\begin{aligned} Q_b |phys\rangle = 0 &\Rightarrow K^{0i} |phys\rangle = 0, \quad K^{ij} |phys\rangle = 0, \quad \varepsilon^{0ijklm} (\partial_k \mathcal{K}_{lm}) |phys\rangle = 0, \\ Q_d |phys\rangle = 0 &\Rightarrow \mathcal{K}^{0i} |phys\rangle = 0, \quad \mathcal{K}^{ij} |phys\rangle = 0, \quad \varepsilon^{0ijklm} (\partial_k \mathcal{K}_{lm}) |phys\rangle = 0. \end{aligned} \quad (84)$$

We note that the above constraint conditions on the physical states are nothing but  $\Pi^{ij} |phys\rangle = 0$ ,  $\Pi^{0i} |phys\rangle = 0$  and  $\dot{\Pi}^{ij} |phys\rangle = 0$ . This can be seen from the equations of motion quoted in (38). It is, furthermore, interesting to note that  $Q_d |phys\rangle = 0$  yields the constraints on the physical states that are *dual* to whatever we obtain from  $Q_b |phys\rangle = 0$ . This can be checked from the discrete symmetry transformations (19) as well.

The other two requirements  $Q_{ab} |phys\rangle = 0$ ,  $Q_{ad} |phys\rangle = 0$  (that emerge from the requirement:  $Q_w |phys\rangle = 0$ ) lead to the same restrictions as (84). Thus, we finally note

that the primary first-class constraints ( $\Pi^{ij} \approx 0, \Pi^{0i} \approx 0$ ) and the time derivative ( $\dot{\Pi}^{ij} \approx 0$ ) annihilate the physical state ( $|phys\rangle$ ) of the theory which emerge from  $Q_b|phys\rangle = 0$ . As a consequence, our quantization scheme is consistent with the Dirac's prescription for the quantization of physical systems with constraints. In our present theory, we also obtain the dual-version of the above constraints on the physical states from  $Q_d|phys\rangle = 0$ . These conditions  $Q_d|phys\rangle = 0, Q_b|phys\rangle = 0$  might force the 6D Abelian 3-form gauge theory, to be a model for the q-TFT. We plan to dwell on this issue in the next section.

## 8 Physical applications of nilpotent symmetries: Interesting observations

In our earlier works on 2D (non-)Abelian 1-form gauge theories [10-12], we have exploited the *on-shell* nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations to prove that these theories are exact models of topological field theories (TFTs). We have been able to express the Lagrangian densities as well as symmetric energy momentum tensors of these 2D theories as the sum of BRST and co-BRST exact terms. Furthermore, we have obtained *four* sets of topological invariants ( $\bar{I}_k$ ) $I_k$  and ( $\bar{J}_k$ ) $J_k$  w.r.t. the *off-shell* nilpotent (anti-)BRST as well as (anti-)co-BRST transformations which obey proper recursion relations (see, e.g. [33] for details). These invariants are succinctly expressed, respectively, as

$$\bar{I}_k = \oint_{C_k} \bar{V}_k, \quad I_k = \oint_{C_k} V_k, \quad \bar{J}_k = \oint_{C_k} \bar{W}_k, \quad J_k = \oint_{C_k} W_k, \quad (85)$$

where  $(\bar{V}_k)V_k$  and  $(\bar{W}_k)W_k$  are the  $k$ -forms ( $k = 0, 1, 2$ ) and  $C_k$  are the homology cycles on the 2D closed Riemann surface (that is to be an Euclidean version of 2D non-compact Minkowskian spacetime manifold) on which the 2D theory is defined. It is essential to have the Euclidean version of the 2D non-compact Minkowskian manifold so that the topological invariants, homology cycles, etc., could find their proper physical/geometrical meaning [33].

As far as the physical application of our present (anti-)BRST and (anti-)co-BRST symmetry transformations is concerned, we would like to study, first of all, the nature of Lagrangian density (30) in terms of the following *on-shell* nilpotent symmetries

$$\begin{aligned} \tilde{s}_b A_{\mu\nu\eta} &= \partial_\mu C_{\nu\eta} + \partial_\nu C_{\eta\mu} + \partial_\eta C_{\mu\nu}, & \tilde{s}_b C_{\mu\nu} &= \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, & \tilde{s}_b \beta_\mu &= \partial_\mu C_2, \\ \tilde{s}_b \bar{C}_{\mu\nu} &= \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}), & \tilde{s}_b \bar{\beta}_\mu &= \frac{1}{2} (\partial^\eta \bar{C}_{\eta\mu} + \partial_\mu \bar{C}_1), \\ \tilde{s}_b \phi_\mu^{(1)} &= \frac{1}{2} (\partial^\eta C_{\eta\mu} + \partial_\mu C_1), & \tilde{s}_b C_1 &= (\partial \cdot \beta), & \tilde{s}_b \bar{C}_2 &= (\partial \cdot \bar{\beta}), \\ \tilde{s}_b \bar{C}_1 &= (\partial \cdot \phi^{(1)}), & \tilde{s}_b [C_2, \phi_\mu^{(2)}, H_{\mu\nu\eta\kappa}] &= 0, \end{aligned} \quad (86)$$

$$\begin{aligned} \tilde{s}_d A_{\mu\nu\eta} &= \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} \partial^\kappa \bar{C}^{\rho\sigma}, & \tilde{s}_d \bar{C}_{\mu\nu} &= \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, & \tilde{s}_d \bar{\beta}_\mu &= \partial_\mu \bar{C}_2, \\ \tilde{s}_d C_{\mu\nu} &= \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} + \frac{1}{2} (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}), & \tilde{s}_d \beta_\mu &= -\frac{1}{2} (\partial^\eta C_{\eta\mu} + \partial_\mu C_1), \\ \tilde{s}_d \phi_\mu^{(2)} &= \frac{1}{2} (\partial^\eta \bar{C}_{\eta\mu} + \partial_\mu \bar{C}_1), & \tilde{s}_d C_2 &= -(\partial \cdot \beta), & \tilde{s}_d \bar{C}_1 &= (\partial \cdot \bar{\beta}), \\ \tilde{s}_d C_1 &= (\partial \cdot \phi^{(2)}), & \tilde{s}_d [\bar{C}_2, \phi_\mu^{(1)}, \partial^\eta A_{\eta\mu\nu}] &= 0, \end{aligned} \quad (87)$$

which are derived from the off-shell nilpotent BRST and co-BRST symmetry transformations [cf. (32), (47)] by the following substitutions

$$\begin{aligned}
K_{\mu\nu} &= \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}), & B_2 &= (\partial \cdot \bar{\beta}), \\
\mathcal{K}_{\mu\nu} &= \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} H^{\eta\kappa\lambda\rho} + \frac{1}{2} (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}), \\
B_1 &= (\partial \cdot \phi^{(1)}), & B_3 &= (\partial \cdot \phi^{(2)}), & B &= -(\partial \cdot \beta), \\
f_\mu &= \frac{1}{2} (\partial^\eta C_{\eta\mu} + \partial_\mu C_1), & \bar{F}_\mu &= \frac{1}{2} (\partial^\eta \bar{C}_{\eta\mu} + \partial_\mu \bar{C}_1).
\end{aligned} \tag{88}$$

The above relations emerge, as the equations of motion, from the Lagrangian density (30). We note that  $\tilde{s}_{(b)d}$  are on-shell ( $\square C_1 = 0, \square \bar{C}_1 = 0, \square C_2 = 0, \square \bar{C}_2 = 0, \square \beta_\mu = 0, \square \bar{\beta}_\mu = 0, \square \phi_\mu^{(1)} + \partial_\mu (\partial \cdot \phi^{(1)}) = 0, \square \phi_\mu^{(2)} + \partial_\mu (\partial \cdot \phi^{(2)}) = 0, \square C_{\mu\nu} - (3/4) [\partial_\mu (\partial^\eta C_{\eta\nu}) - \partial_\nu (\partial^\eta C_{\eta\mu})] = 0, \square \bar{C}_{\mu\nu} - (3/4) [\partial_\mu (\partial^\eta \bar{C}_{\eta\nu}) - \partial_\nu (\partial^\eta \bar{C}_{\eta\mu})] = 0$ ) nilpotent of order two ( $\tilde{s}_{(b)d}^2 = 0$ ). Furthermore, these are actual symmetry transformations for the following Lagrangian density

$$\begin{aligned}
\tilde{\mathcal{L}}_{(b)} &= \frac{1}{2} \left[ \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}) \right] \left[ \partial_\kappa A^{\kappa\mu\nu} + \frac{1}{2} (\partial^\mu \phi^{\nu(1)} - \partial^\nu \phi^{\mu(1)}) \right] \\
&- \frac{1}{2} \left[ \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} + \frac{1}{2} (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}) \right] \left[ \frac{1}{4!} \varepsilon^{\mu\nu\alpha\gamma\lambda\zeta} H_{\alpha\gamma\lambda\zeta} + \frac{1}{2} (\partial^\mu \phi^{\nu(2)} - \partial^\nu \phi^{\mu(2)}) \right] \\
&+ \left( \partial_\mu \bar{C}_{\nu\eta} + \partial_\nu \bar{C}_{\eta\mu} + \partial_\eta \bar{C}_{\mu\nu} \right) (\partial^\mu C^{\nu\eta}) - (\partial \cdot \bar{\beta}) (\partial \cdot \beta) - \partial_\mu \bar{C}_2 \partial^\mu C_2 + \frac{1}{2} (\partial \cdot \phi^{(1)})^2 \\
&- \frac{1}{2} (\partial \cdot \phi^{(2)})^2 + \frac{1}{2} (\partial_\mu \bar{C}^{\mu\nu} + \partial^\nu \bar{C}_1) (\partial^\eta C_{\eta\nu} + \partial_\nu C_1) - (\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu) (\partial^\mu \beta^\nu),
\end{aligned} \tag{89}$$

which is derived from (30) by the appropriate substitutions from (88). For an exact topological field theory, it is essential that this Lagrangian density should be able to be expressed as the sum of BRST and co-BRST exact terms (especially for a field theoretic model for the Hodge theory as is the case with our earlier works on 2D (non)Abelian theories [10]).

We demonstrate that our present model of 6D Abelian 3-form gauge theory is a model for a quasi-topological field theory (q-TFT) because one can express (89) as the sum of BRST and co-BRST exact terms modulo a total spacetime derivative term plus a single *extra* term. To corroborate the above statement, it can be checked that, we have

$$\begin{aligned}
\tilde{\mathcal{L}}_{(b)} &= \tilde{s}_b [T_1 + T_2 + T_3 + T_4 + T_5] + \tilde{s}_d [P_1 + P_2 + P_3 + P_4 + P_5] \\
&- \frac{1}{2} (\partial_\mu \bar{C}^{\mu\nu}) (\partial^\eta C_{\eta\nu}) - \partial_\mu Z^\mu.
\end{aligned} \tag{90}$$

Here the exact expressions for  $T_i$ ,  $P_i$  ( $i = 1, 2, \dots, 5$ ) and  $Z^\mu$  are

$$\begin{aligned}
T_1 &= \frac{1}{2} \left[ \partial^\eta A_{\eta\mu\nu} + \frac{1}{2} (\partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}) \right] \bar{C}^{\mu\nu}, & T_2 &= \frac{1}{2} (\partial \cdot \phi^{(1)}) \bar{C}_1, \\
T_3 &= -\frac{1}{2} (\partial \cdot \beta) \bar{C}_2, & T_4 &= -\frac{1}{2} (\partial_\mu \bar{\beta}_\nu) C^{\mu\nu}, & T_5 &= -\frac{1}{2} (\partial \cdot \bar{\beta}) C_1,
\end{aligned} \tag{91}$$

$$\begin{aligned}
P_1 &= -\frac{1}{2} \left[ \frac{1}{4!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} H^{\eta\kappa\rho\sigma} + \frac{1}{2} (\partial_\mu \phi_\nu^{(2)} - \partial_\nu \phi_\mu^{(2)}) \right] C^{\mu\nu}, & P_2 &= -\frac{1}{2} (\partial \cdot \phi^{(2)}) C_1, \\
P_3 &= \frac{1}{2} (\partial \cdot \bar{\beta}) C_2, & P_4 &= -\frac{1}{2} (\partial_\mu \beta_\nu) \bar{C}^{\mu\nu}, & P_5 &= \frac{1}{2} (\partial \cdot \beta) \bar{C}_1,
\end{aligned} \tag{92}$$

$$\begin{aligned}
Z^\mu &= \frac{1}{2} \left[ \left( \partial^\mu C^{\nu\eta} + \partial^\nu C^{\eta\mu} + \partial^\eta C^{\mu\nu} \right) \bar{C}_{\nu\eta} - (\partial_\eta \bar{C}^{\eta\mu} + \partial^\mu \bar{C}_1) C_1 \right. \\
&\quad - \left( \partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta\mu} + \partial^\eta \bar{C}^{\mu\nu} \right) C_{\nu\eta} - (\partial^\mu C_2) \bar{C}_2 + (\partial^\mu \bar{C}_2) C_2 \\
&\quad \left. - (\partial^\eta \bar{C}_{\eta\nu} + \partial_\nu \bar{C}_1) C^{\mu\nu} - \bar{C}^{\mu\nu} (\partial^\eta C_{\eta\nu}) \right].
\end{aligned} \tag{93}$$

Thus, we note that the Lagrangian density (90) looks very much like the model for an exact TFT *but* for the term  $\frac{1}{2} (\partial_\mu \bar{C}^{\mu\nu}) (\partial^\eta C_{\eta\nu})$ . This is why we have christened our present theory as a model for the q-TFT because it misses by a single term to be an exact model for TFT. The same kind of observation has been made in our earlier work on 4D free Abelian 2-form gauge theory where we have proven its quasi-topological nature [24].

We now focus on the topological invariants like (85). For our present 6D Abelian 3-form gauge theory, the invariants like (85), are as follows

$$\begin{aligned}
V_0(+3) &= B_1 C_2, \\
V_1(+2) &= \left[ (\partial_\mu \bar{C}_1) C_2 + B_1 \beta_\mu \right] dx^\mu, \\
V_2(+1) &= \left[ (\partial_\mu \bar{C}_1) \beta_\nu + \frac{1}{2!} B_1 C_{\mu\nu} \right] (dx^\mu \wedge dx^\nu), \\
V_3(0) &= \left[ \frac{1}{2!} (\partial_\mu \bar{C}_1) C_{\nu\eta} + \frac{1}{3!} B_1 A_{\mu\nu\eta} \right] (dx^\mu \wedge dx^\nu \wedge dx^\eta), \\
V_4(-1) &= \left[ \frac{1}{3!} (\partial_\mu \bar{C}_1) A_{\nu\eta\kappa} + \frac{1}{4!} \bar{C}_1 H_{\mu\nu\eta\kappa} \right] (dx^\mu \wedge dx^\nu \wedge dx^\eta \wedge dx^\kappa) \\
&\equiv d \left[ \frac{1}{3!} \bar{C}_1 A_{\nu\eta\kappa} \right] (dx^\nu \wedge dx^\eta \wedge dx^\kappa), \\
V_5(-2) &= 0, \quad (d^2 = 0).
\end{aligned} \tag{94}$$

The above invariants are defined (w.r.t. BRST charge) on the Euclidean version of the non-compact 6D Minkowskian spacetime manifold on which our present theory is considered. At this stage, a few useful comments are in order. First, the zero-form invariant  $V_0(+3)$  is BRST invariant (i.e.  $s_b V_0 = 0$ ). Second, the numbers  $(+3, +2, +1, 0, -1)$  in the round brackets correspond to the ghost numbers that uniquely characterize the topological invariants. Third, the invariants terminate at  $k = 4$  as,  $V_4(-1)$  with ghost number  $(-1)$ , turns out to be an exact form. Fourth, the topological invariants  $\bar{I}_k$  (w.r.t. anti-BRST symmetries) can be obtained from  $I_k$  ( $k = 0, 1, 2, 3, 4, 5$ ) by the replacements:  $C_2 \rightarrow \bar{C}_2$ ,  $C_{\mu\nu} \rightarrow \bar{C}_{\mu\nu}$ ,  $K_{\mu\nu} \rightarrow \bar{K}_{\mu\nu}$ ,  $\bar{C}_{\mu\nu} \rightarrow C_{\mu\nu}$ ,  $A_{\mu\nu\eta} \rightarrow \bar{A}_{\mu\nu\eta}$  and  $H_{\mu\nu\eta\kappa} \rightarrow \bar{H}_{\mu\nu\eta\kappa}$ . Fifth, it is clear that the ghost numbers for the existing invariants  $\bar{I}_k$  w.r.t. anti-BRST symmetries would be in the order  $(-3, -2, -1, 0, +1)$ , respectively. Finally, as a key signature of the

topological properties, the above invariants obey the following recursion relations:

$$s_b I_k = d I_{k-1}, \quad s_{ab} \bar{I}_k = d \bar{I}_{k-1}, \quad k = 1, 2, 3, 4, 5. \quad (95)$$

Thus, it is clear that our present model of 6D Abelian 3-form gauge theory captures one of the key features of an exact TFT (see, e.g. [10-12] for details).

Analogous to the topological invariants  $(\bar{I}_k)I_k$  w.r.t. off-shell nilpotent symmetries and corresponding conserved (anti-)BRST charges, one can write down the invariants with respect to the (anti-)co-BRST charges. The ones, w.r.t. the co-BRST charge, are

$$\begin{aligned} \bar{W}_0(-3) &= B_3 \bar{C}_2, \\ \bar{W}_1(-2) &= \left[ (\partial_\mu C_1) \bar{C}_2 + B_3 \bar{\beta}_\mu \right] dx^\mu, \\ \bar{W}_2(-1) &= \left[ (\partial_\mu C_1) \bar{\beta}_\nu + \frac{1}{2!} B_3 \bar{C}_{\mu\nu} \right] (dx^\mu \wedge dx^\nu), \\ \bar{W}_3(0) &= \left[ \frac{1}{3!} (\partial_\mu \bar{C}_1) C_{\nu\eta} + B_3 \left( \frac{1}{3!} \varepsilon_{\mu\nu\eta\kappa\rho\sigma} A^{\kappa\rho\sigma} \right) \right] (dx^\mu \wedge dx^\nu \wedge dx^\eta), \\ \bar{W}_4(+1) &= \left[ \frac{1}{3!} (\partial_\mu C_1) \left( \frac{1}{3!} \varepsilon_{\nu\eta\kappa\lambda\rho\sigma} A^{\lambda\rho\sigma} \right) \right. \\ &\quad \left. + \frac{1}{3!} B_3 \left( \frac{1}{3!} \varepsilon_{\nu\eta\kappa\lambda\rho\sigma} \partial_\mu A^{\lambda\rho\sigma} \right) \right] (dx^\mu \wedge dx^\nu \wedge dx^\eta \wedge dx^\kappa) \\ &\equiv d \left[ \frac{1}{3!} C_1 \left( \frac{1}{3!} \varepsilon_{\nu\eta\kappa\lambda\rho\sigma} A^{\lambda\rho\sigma} \right) \right] (dx^\nu \wedge dx^\eta \wedge dx^\kappa), \\ \bar{W}_5(+2) &= 0, \quad (d^2 = 0). \end{aligned} \quad (96)$$

A few comments are in order, at this juncture. First, it can be checked that  $\bar{W}_0(-3)$  is a co-BRST invariant quantity [i.e.  $s_d \bar{W}_0(-3) = 0$ ]. Second, the *five* numbers  $(-3, -2, -1, 0, +1)$  in the round brackets correspond to the ghost numbers which provide the accurate characterization of a specific invariant. Third, one can obtain the k-forms  $W_k$  ( $k = 0, 1, 2, 3, 4$ ) w.r.t. anti-co-BRST charge by the replacements:  $\bar{C}_2 \rightarrow C_2$ ,  $\bar{C}_{\mu\nu} \rightarrow C_{\mu\nu}$ ,  $\bar{K}_{\mu\nu} \rightarrow \bar{K}_{\mu\nu}$ ,  $C_{\mu\nu} \rightarrow \bar{C}_{\mu\nu}$ ,  $\bar{\beta}_\mu \rightarrow \beta_\mu$  and  $A_{\mu\nu\eta} \rightarrow A_{\mu\nu\eta}$ . Fourth, the ghost numbers of invariants  $W_k$  (with  $k = 0, 1, 2, 3, 4, 5$ ) would be in the order  $(+3, +2, +1, 0, -1)$ . Fifth, the forms  $\bar{W}_4(+1)$  and  $W_4(-1)$  turn out to be exact forms. As a consequence, we find that  $\bar{W}_5 = 0, W_5 = 0$  due to  $d^2 = 0$ . Finally, the above invariants follow the recurrence relations

$$s_d \bar{W}_k = d \bar{W}_{k-1}, \quad s_{ad} W_k = d W_{k-1}, \quad k = 1, 2, 3, 4, 5. \quad (97)$$

Thus, we note that our present theory does capture one of the key features of TFT. We have purposely denoted the invariants w.r.t. the co-BRST charge with a bar ( $\bar{W}_k$ ) because the anti-ghost fields appear in the co-BRST symmetry transformations [cf. (87)]. As a consequence, the invariants, w.r.t. the anti-co-BRST charge, are denoted without a bar. This observation should be contrasted with the invariants w.r.t. BRST and anti-BRST charges where we have taken the opposite convention for the notations of these invariants.

Now we focus on the invariants starting with the ghost number  $(+2)$  w.r.t. the BRST

charge. A set of such quantities [that follows the recursion relations (95)] are

$$\begin{aligned}
V_0(+2) &= K^{\mu\nu}(\partial_\mu\beta_\nu), \\
V_1(+1) &= \left[ (\partial_\mu\bar{C}^{\nu\eta})(\partial_\nu\beta_\eta) + \frac{1}{2!} K^{\nu\eta}(\partial_\mu C_{\nu\eta}) \right] dx^\mu, \\
V_2(0) &= \left[ \frac{1}{2!} (\partial_\mu\bar{C}^{\eta\kappa})(\partial_\nu C_{\eta\kappa}) \right] (dx^\mu \wedge dx^\nu) \\
&\equiv d \left[ \frac{1}{2!} \bar{C}^{\eta\kappa}(\partial_\nu C_{\eta\kappa}) \right] dx^\nu, \\
V_3(-1) &= 0, \quad (d^2 = 0).
\end{aligned} \tag{98}$$

As discussed earlier, we can obtain the invariants starting with ghost number  $(-2)$  w.r.t. the conserved and nilpotent anti-BRST charge by the replacements:  $K_{\mu\nu} \rightarrow \bar{K}_{\mu\nu}$ ,  $C_{\mu\nu} \rightarrow \bar{C}_{\mu\nu}$ ,  $\bar{C}_{\mu\nu} \rightarrow C_{\mu\nu}$  and  $\beta_\mu \rightarrow \bar{\beta}_\mu$ . To compute the invariants  $(\bar{W}_k)$  (with  $k = 0, 1, 2, 3$ ) w.r.t. the co-BRST charge, we have to replace:  $K_{\mu\nu} \rightarrow \mathcal{K}_{\mu\nu}$ ,  $\beta_\mu \rightarrow \bar{\beta}_\mu$ ,  $C_{\mu\nu} \rightarrow \bar{C}_{\mu\nu}$ ,  $\bar{C}_{\mu\nu} \rightarrow C_{\mu\nu}$  in the above set of invariants. As is evident, these invariants will be characterized by the ghost numbers  $(-2, -1, 0)$ . These invariants, w.r.t. the nilpotent (anti-)co-BRST charges, obey the recursion relations (97). Similarly, the invariants  $W_k$  (with  $k = 0, 1, 2, 3$ ), w.r.t. the anti-co-BRST charge, can be obtained from  $V_k$  by the *only* the replacement  $K_{\mu\nu} \rightarrow \bar{K}_{\mu\nu}$ . As a consequence, these invariants would be characterized by the ghost numbers  $(+2, +1, 0)$ . Thus, we conclude that if we know the set of invariants w.r.t. the conserved and nilpotent BRST charge, we can obtain all the other invariants w.r.t. anti-BRST and (anti-)co-BRST charges by exploiting the appropriate discrete symmetry transformations listed in Sec. VI.

We venture, now, to obtain the invariants w.r.t. the nilpotent and conserved BRST charge corresponding to the ghost number  $(+1)$ . These invariants, obeying the recursion relations (95), are as follows

$$\begin{aligned}
V_0(+1) &= K^{\mu\nu}C_{\mu\nu} - \bar{C}^{\mu\nu}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu), \\
V_1(0) &= \left[ (\partial_\mu\bar{C}^{\nu\eta})C_{\nu\eta} + \bar{C}^{\nu\eta}(\partial_\mu C_{\nu\eta}) \right] dx^\mu \equiv d \left[ \bar{C}^{\nu\eta}C_{\nu\eta} \right], \\
V_2(-1) &= 0, \quad (d^2 = 0).
\end{aligned} \tag{99}$$

It is evident, from our previous discussions, that the invariants [starting with the ghost number  $(-1)$ ] w.r.t. the anti-BRST charge can be obtained from the above by the replacements:  $K_{\mu\nu} \rightarrow \bar{K}_{\mu\nu}$ ,  $C_{\mu\nu} \rightarrow \bar{C}_{\mu\nu}$ ,  $\bar{C}_{\mu\nu} \rightarrow C_{\mu\nu}$ ,  $\beta_\mu \rightarrow \bar{\beta}_\mu$ . These invariants would also obey the recursion relations (95). From the set of invariants listed in (99), we can obtain all the invariants of theory w.r.t. anti-BRST and (anti-)co-BRST charges by appropriate use of discrete symmetry transformation of Sec. VI. This has already been done for the invariants starting with ghost number  $(+2)$  from equation (98). Exactly the same substitutions, from the discrete symmetry transformations of Sec. VI, have to be exploited here, too.

We make, in the following, some remarks that are very decisive. These are connected with some invariants which obey exactly the same recursion relations as (95) and (97) but they are *not* physically interesting on various grounds. First, we discuss about the alternative to the BRST invariants with the characteristic ghost number  $(+1)$ . These can

be also constructed as follows

$$\begin{aligned}
V_0(+1) &= (\partial_\mu K^{\mu\nu}) f_\nu, \\
V_1(0) &= \left[ \partial_\mu (\partial_\nu \bar{C}^{\nu\eta}) f_\eta + (\partial_\nu K^{\nu\eta}) (\partial_\mu \phi_\eta^{(1)}) \right] dx^\mu, \\
V_2(-1) &= \left[ \partial_\mu (\partial_\eta \bar{C}^{\eta\kappa}) (\partial_\nu \phi_\kappa^{(1)}) \right] dx^\mu \wedge dx^\nu \equiv d \left[ (\partial_\eta \bar{C}^{\eta\kappa}) (\partial_\nu \phi_\kappa^{(1)}) \right] dx^\nu, \\
V_3(-2) &= 0, \quad (d^2 = 0).
\end{aligned} \tag{100}$$

However, one can check that the mass dimension of  $V_0(+1)$ , in the above equation, is *seven* (i.e.  $[V_0(+1)] = [M]^7$ ) in the natural units where  $\hbar = c = 1$ . In a 6D theory, an invariant of mass dimension *seven* is *not* allowed. Thus, we do not include (100) in the list of appropriate invariants. Similarly, a set of BRST invariants starting with the ghost number (+2) can be constructed as follows:

$$\begin{aligned}
V_0(+2) &= (\partial \cdot \bar{F}) C_2, \\
V_1(+1) &= \left[ \partial_\mu (\partial \cdot \bar{\beta}) C_2 - (\partial \cdot \bar{F}) \beta_\mu \right] dx^\mu, \\
V_2(0) &= \left[ \frac{1}{2!} (\partial \cdot \bar{F}) C_{\mu\nu} - \partial_\mu (\partial \cdot \bar{\beta}) \beta_\nu \right] (dx^\mu \wedge dx^\nu), \\
V_3(-1) &= \left[ \frac{1}{2!} \partial_\mu (\partial \cdot \bar{\beta}) C_{\nu\eta} - \frac{1}{3!} (\partial \cdot \bar{F}) A_{\mu\nu\eta} \right] (dx^\mu \wedge dx^\nu \wedge dx^\eta), \\
V_4(-2) &= - \left[ \frac{1}{3!} \partial_\mu (\partial \cdot \bar{\beta}) A_{\nu\eta\kappa} + \frac{1}{4!} (\partial \cdot \bar{\beta}) H_{\mu\nu\eta\kappa} \right] (dx^\mu \wedge dx^\nu \wedge dx^\eta \wedge dx^\kappa) \\
&\equiv d \left[ -\frac{1}{3!} (\partial \cdot \bar{\beta}) A_{\nu\eta\kappa} \right] (dx^\nu \wedge dx^\eta \wedge dx^\kappa), \\
V_5(-3) &= 0, \quad (d^2 = 0),
\end{aligned} \tag{101}$$

which obey the correct recursion relation. As discussed earlier, other invariants corresponding to (100) and (101) can also be obtained by exploiting the appropriate discrete symmetry transformations of Sec. VI. We note that the mass dimension of  $V_0(+2)$  is seven (i.e.  $[V_0(+2)] = [M]^7$ ) in natural units (where  $\hbar = c = 1$ ). This kinds of invariants are *not* allowed in a 6D theory where the maximum physical dimensions of BRST-invariant quantities can be at most six. There is another argument in favour of discarding (101) as physical invariants. We note that only ghost fields are present in (101) which are not physical fields. Thus, above set of invariants (101) are *not* physically interesting.

We wrap up this section with a couple of remarks. First, we have discussed the quasi-topological nature of 6D Abelian 3-form gauge theory by exploiting the on-shell nilpotent BRST and co-BRST symmetry transformations [cf. (90)]. However, one can discuss the above properties in terms of the on-shell nilpotent anti-BRST and anti-co-BRST symmetry as well (where the Lagrangian density (31) would play very important role). Second, it is known in literature that, even if there are propagating degrees of freedom in a theory, the theory can still capture some of the key features of TFT (see, e.g. [34-35] for details). We have shown that, exactly above kind of situation is prevalent in the case of 4D Abelian 2-form [24] as well as 6D Abelian 3-form gauge theories which are perfect models for the

Hodge theory. This observation should be contrasted with the 2D Abelian 1-form gauge theory which is a perfect model for the exact TFT as well as Hodge theory [10]. In all the above claims, the dual-BRST (i.e. co-BRST) symmetry plays a very significant role.

## 9 Conclusions

In our present endeavor, we have been able to establish that the “classical” dual-gauge symmetry transformations would be always associated with any arbitrary Abelian  $p$ -form ( $p = 1, 2, 3, \dots$ ) gauge theory in some specific  $D$ -dimensions of spacetime (when  $D = 2p$ ). We have been able to show that this is true in the cases of Abelian 1-form gauge theory in two  $(1 + 1)$ -dimensions of spacetime, Abelian 2-form gauge theory in four  $(3 + 1)$ -dimensions of spacetime and, in our present investigation, we have shown the existence of dual-gauge symmetry transformations for the Abelian 3-form gauge theory in higher ( $D > 4$ ) six  $(5 + 1)$ -dimensions of spacetime in great detail. In all the above discussions, we have considered, for the sake of simplicity, the gauge-fixed Lagrangian densities in the Feynman gauge *only*. However, our discussions could be easily generalized to the gauge-fixed Lagrangian densities, of the above gauge theories, in any arbitrary gauge. Thus, we firmly believe that a  $D = 2p$  dimensional Abelian  $p$ -form gauge theory would *always* be endowed with the dual-gauge symmetry transformations at the “classical” level.

As is well-known and firmly well-established, the *usual* gauge symmetries are generated by the first-class constraints (in the language of Dirac’s prescription [31,32] for the classification scheme) of any arbitrary  $p$ -form gauge theory in any arbitrary  $D$ -dimensions of spacetime. However, the dual-gauge symmetries exist for any arbitrary Abelian  $p$ -form gauge theory *only* in  $D = 2p$  dimensions of spacetime where the origin for such an existence lies in the *self-duality* condition [cf. (4),(8),(15)]. In other words, the dual-gauge symmetry exists for the Abelian  $p$ -form gauge theories where there is a self-duality condition which is mathematically dictated by the Levi-Civita tensor of the  $D = 2p$  dimensions of spacetime. There is yet another distinguishing feature that differentiates the gauge- and dual-gauge symmetry transformations for an Abelian theory. Whereas the curvature tensor (owing its origin to the exterior derivative) remains invariant under the continuous gauge symmetry transformations, it is the gauge-fixing term (owing its origin to the co-exterior derivative) that remains invariant under the continuous dual-gauge symmetry transformations.

For the  $2p$ -dimensional Abelian  $p$ -form gauge theories, one can generalize the above “classical” dual-gauge symmetry transformations to the “quantum” level in the language of off-shell nilpotent (anti-)dual BRST [or (anti-)co-BRST] symmetry transformations. The latter off-shell nilpotent and absolutely anticommuting symmetry transformations should be contrasted with the usual off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations. In fact, it is the *total* kinetic terms that remain invariant under the proper (anti-)BRST symmetry transformations whereas it is the *total* gauge-fixing terms that remain unchanged under the proper (anti-)co-BRST symmetry transformations. For the Abelian  $p$ -form gauge theory, the curvature tensor for the gauge field (owing its origin to the exterior derivative) remains *certainly* invariant under the off-shell nilpotent (anti-)BRST transformations. However, for such theories, it is the gauge-fixing term for the same gauge field (owing its origin to the co-exterior derivative) that remains *definitely* unchanged



under the (anti-)co-BRST transformations. Hence, the nomenclatures are very appropriate for the above nilpotent symmetry transformations.

It is of utmost importance to point out that *exactly* similar kind of restrictions [cf. (3), (6), (13)] must be imposed on the (dual-)gauge parameters for the (dual-)gauge invariance of the gauge-fixed Lagrangian densities of any arbitrary  $D = 2p$  dimensional Abelian  $p$ -form gauge theory. Within the framework of the BRST formalism, however, there are *no* such restrictions on any parameters of the theory. This is due to the fact that the equations of motion for the Faddeev-Popov (anti-)ghost fields of the theory take care of these restrictions that are found for the (dual-)gauge invariance of the gauge-fixed Lagrangian densities. The coupled Lagrangian densities (30), (31) respect both the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations [cf. (34), (32), (49) and (47)] on the hypersurface defined by the CF-type restrictions [cf. (24) and (43)].

For all the  $D = 2p$  dimensional Abelian  $p$ -form gauge theories, one can define a bosonic symmetry in the theory that emerges from the suitable anticommutators between the above nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations. The above nilpotent (anti-)co-BRST and (anti-)BRST symmetry transformations provide the physical realizations of the (co-)exterior derivatives and their appropriate anticommutators provide the physical realization of the Laplacian operator. There always exists a ghost-scale symmetry in the above theories which is needed for the definition of the ghost number of a state in the quantum Hilbert space. Together, the algebraic structures of all the six continuous symmetries (and their corresponding generators) provide the physical realizations of the algebra obeyed by the de Rham cohomological operators (as well as the Hodge decomposition theorem that is defined in terms of the above cohomological operators [cf. (80)-(82)]).

There exists a discrete set of symmetries in the above  $D = 2p$  dimensional Abelian  $p$ -form gauge theories that is connected with the self-duality conditions [cf. (4), (8), (15)]. This condition, in turn, is very intimately related with the Hodge duality (\*) operation of differential geometry. As a consequence, these symmetries provide the physical realization of the Hodge duality (\*) operation (of differential geometry). Thus, the spacetime  $D = 2p$  dimensional Abelian  $p$ -form gauge theories automatically provide the field theoretic models for the Hodge theory where all the de Rham cohomological operators, Hodge duality (\*) operation and Hodge decomposition theorem, etc., find their physical realizations in the language of discrete and continuous symmetry properties of these theories. Furthermore, it turns out that the mathematical condition  $* d A^{(p)} = \delta A^{(p)}$  is always satisfied for the Abelian  $p$ -form gauge theories in  $D = 2p$  dimensions of flat spacetime. Such theories are always endowed with the dual-gauge and off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetry transformations (see, e.g., Appendix D for details).

One of the novel observations in our present investigation is the fact that  $\{s_b, s_{ad}\} = s_\omega^{(1)}$  and  $\{s_d, s_{ab}\} = s_\omega^{(2)}$  define *new* bosonic symmetries in the theory. However, these new bosonic symmetries cannot be identified with the Laplacian operator of differential geometry because these do *not* commute with the ghost-scale symmetry transformations. Hence, these do not correspond to the Casimir operators for the whole algebra. Further, a close look and a careful observation of the transformations in (67) shows that  $s_d$  and  $s_{ab}$  (as well as  $s_b$  and  $s_{ad}$ ) anticommute with each-other up to a U(1) vector gauge transformations. In the context of 2D Abelian 1-form [10-12] and 4D Abelian 2-form gauge theory [13-16], it

has been found that  $\{s_b, s_{ad}\} = 0$  and  $\{s_d, s_{ab}\} = 0$  which imply the absolute anticommutativity between the above nilpotent (fermionic) transformations. We have discussed, in detail, about these new bosonic symmetries in our Appendix E and established that these can *not* be identified with the Laplacian operator of differential geometry (due to the ghost number considerations and their connection with the degree of a form).

In our earlier works on 2D Abelian 1-form [10] and 4D Abelian 2-form gauge theories [24], we have shown that the former theories turn out to be a perfect model for a new TFT and the latter theory is proven to be a model for q-TFT. Both varieties of theories are, however, perfect field theoretic models for the Hodge theory. We have proven, in our present investigation (cf. Sec. VIII) that the 6D Abelian 3-form gauge theory, besides being a perfect model for the Hodge theory, is also a model for the q-TFT where the physical application of the dual-BRST symmetry turns up in a very convincing and cogent manner. In other words, as is evident from the d.o.f. counting, we have demonstrated that the 6D Abelian 3-form gauge theory is *not* a perfect model for the TFT. We note, in passing, that the physical contents of the 4D Abelian 2-form and 6D Abelian 3-form gauge theories are very much similar to each-other within the framework of BRST formalism as both of them are the perfect models for the Hodge theory as well as q-TFT.

We have not discussed, in our present investigation, anything about the interacting  $p$ -form gauge theories (with matter fields) as well as the more general non-Abelian  $p$ -form gauge theories. In this connection, we would like to state that, so far, we have been able to establish that the interacting 2D Abelian 1-form gauge theory (with Dirac fields) and the 2D (non-)Abelian 1-form gauge theories (without any interaction with matter fields) also provide the field theoretic models for the Hodge theory (see, e.g., [11,12]). It would be very challenging endeavor to obtain the tractable field theoretic models for the Hodge theory in the cases of *interacting* (non-)Abelian  $p$ -form gauge theories in any arbitrary dimensions of spacetime where matter fields are also present. These are some of the issues that are presently under investigation and our results would be reported in the future.

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## Appendix A

In our earlier work [22], we have derived the gauge-fixing and Faddeev-Popov ghost terms for the starting Lagrangian density ( $\mathcal{L}_{(0)} = \frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa}$ ), within the framework of BRST formalism, by exploiting the following standard expressions (see, e.g. [21,22]):

$$\begin{aligned} \mathcal{L}_{(b)} = & \frac{1}{2} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa} + s_b s_{ab} \left[ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_{\mu\nu} C^{\mu\nu} - \frac{1}{2} \phi_\mu^{(1)} \phi^{\mu(1)} \right. \\ & \left. - \frac{1}{2} \phi_\mu^{(2)} \phi^{\mu(2)} - \bar{\beta}_\mu \beta^\mu - \frac{1}{6} A_{\mu\nu\eta} A^{\mu\nu\eta} \right], \\ \mathcal{L}_{(\bar{b})} = & \frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa} - s_{ab} s_b \left[ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_{\mu\nu} C^{\mu\nu} - \frac{1}{2} \phi_\mu^{(1)} \phi^{\mu(1)} \right] \end{aligned} \quad (A.1)$$

$$- \frac{1}{2} \phi_\mu^{(2)} \phi^{\mu(2)} - \bar{\beta}_\mu \beta^\mu - \frac{1}{6} A_{\mu\nu\eta} A^{\mu\nu\eta} \Big], \quad (\text{A.2})$$

where  $s_b$  and  $s_{ab}$  are the off-shell nilpotent symmetry transformations (32) and (34), respectively, and  $(\mathcal{L}_{(b)}, \mathcal{L}_{(\bar{b})})$  are the coupled (but equivalent) Lagrangian densities that respect the (anti-)BRST symmetry transformations  $s_{(a)b}$ . Within the above square brackets in (A.1) and (A.2), we have chosen the combinations of fields that have mass dimensions equal to four and ghost numbers equal to zero for the derivation of the 6D (anti-)BRST invariant Lagrangian densities. The emerging Lagrangian densities, however, do *not* produce the fermionic CF-type of conditions (43) from the equations of motion. Thus, we have altered a bit the above combinations of fields in the square brackets of (A.1) and (A.2) as

$$\begin{aligned} \mathcal{L}_{(b)} = & \frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa} + s_b s_{ab} \left[ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_{\mu\nu} C^{\mu\nu} - \frac{1}{2} \phi_\mu^{(1)} \phi^{\mu(1)} \right. \\ & \left. - \frac{1}{2} \phi_\mu^{(2)} \phi^{\mu(2)} + \bar{\beta}_\mu \beta^\mu - \frac{1}{6} A_{\mu\nu\eta} A^{\mu\nu\eta} \right], \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \mathcal{L}_{(\bar{b})} = & \frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa} - s_{ab} s_b \left[ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_{\mu\nu} C^{\mu\nu} - \frac{1}{2} \phi_\mu^{(1)} \phi^{\mu(1)} \right. \\ & \left. - \frac{1}{2} \phi_\mu^{(2)} \phi^{\mu(2)} + \bar{\beta}_\mu \beta^\mu - \frac{1}{6} A_{\mu\nu\eta} A^{\mu\nu\eta} \right], \end{aligned} \quad (\text{A.4})$$

which leads to the derivation of appropriate coupled and equivalent Lagrangian densities (30) and (31) (see, Sec. III) after linearizations of the kinetic term  $(\frac{1}{24} H^{\mu\nu\eta\kappa} H_{\mu\nu\eta\kappa})$  by invoking the Nakanishi-Lautrup type auxiliary fields  $\mathcal{K}_{\mu\nu}$  and  $\bar{\mathcal{K}}_{\mu\nu}$  and introducing the Lorentz vector fields  $\phi_\mu^{(2)}$  [cf. (20),(21)]. The Lagrangian densities [i.e. (30), (31)] yield CF-type of restrictions (24) and (43) from equations of motion [cf. (38), (39)].

We would like to lay stress on the fact that the difference between (A.1) [as well as (A.2)] and (A.3) [as well as (A.4)] is *only* the term  $\bar{\beta}^\mu \beta_\mu$  which carries  $(-)$  sign in the former pair and  $(+)$  sign in the latter pair. The key advantage of the choice in (A.3) [and (A.4)] leans heavily on the derivation of CF-type restrictions (24) and (43) which automatically turn out to be (anti-)BRST as well as (anti-)co-BRST invariant. This happens because (24) and (43) are derived from the equations of motion which always respect the basic symmetries of the theory. As a consequence, these (anti-)BRST and (anti-)co-BRST invariant restrictions are *physical* in some sense (in the light of our theory being a model for the Hodge theory).

## Appendix B

In this Appendix, we shall show that the BRST charge  $Q_b$  is nilpotent of order two (i.e.  $Q_b^2 = 0 \implies \frac{1}{2} \{Q_b, Q_b\} = 0$ ) from the symmetry principle  $s_b Q_b = -i \{Q_b, Q_b\} = 0$  where the conserved and nilpotent BRST charge  $Q_b$  plays the key role of a generator. For this purpose, we use here the BRST symmetry transformations  $s_b$  [cf. (32)] and the expression for the conserved charge  $Q_b$  [cf. (41)]. It can be explicitly checked that

$$s_b Q_b = - \int d^5 x \left[ (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) (\partial_i C_2) + (\partial^0 K^{\nu\eta} + \partial^\nu K^{\eta 0} + \partial^\eta K^{0\nu}) (\partial_\nu \beta_\eta - \partial_\eta \beta_\nu) \right]. \quad (\text{B.1})$$

The following Euler-Lagrange equations of motion [cf. (38)]

$$\begin{aligned} \square \bar{C}_{\mu\nu} - \frac{3}{2}(\partial_\mu \bar{F}_\nu - \partial_\nu \bar{F}_\mu) &= 0, \quad \partial_\mu \bar{C}^{\mu\nu} + \partial^\nu \bar{C}_1 - 2\bar{F}^\nu = 0, \\ \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu \mathcal{K}^{\nu\eta}) &= \partial_\kappa K_{\lambda\rho} + \partial_\lambda K_{\rho\kappa} + \partial_\rho K_{\kappa\lambda}, \end{aligned} \quad (B.2)$$

can be exploited to re-express (B.1) in a different form. For instance, we obtain the following

$$\begin{aligned} (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) &= \frac{2}{3} \square \bar{C}^{0i}, \quad \partial_i \bar{C}^{0i} = \partial^0 \bar{C}_1 - 2\bar{F}^0, \\ \partial^0 K^{\nu\eta} + \partial^\nu K^{\eta 0} + \partial^\eta K^{0\nu} &= -\frac{1}{2} \varepsilon^{0\mu\lambda\rho\nu\eta} (\partial_\mu \mathcal{K}_{\lambda\rho}), \end{aligned} \quad (B.3)$$

from (B.2) as special cases. Using the above equations of motion (B.3) in equation (B.1) and performing a partial integration, we can re-express (B.1) as follows:

$$\begin{aligned} s_b Q_b &= \int d^5x \partial_i \left[ -\frac{2}{3} (\square \bar{C}^{0i}) C_2 + \frac{1}{2} \varepsilon^{0ijklm} \mathcal{K}_{jk} (\partial_l \beta_m - \partial_m \beta_l) \right] \\ &+ \frac{2}{3} \int d^5x \left[ \partial^0 (\square \bar{C}_1) C_2 - 2(\square \bar{F}^0) C_2 \right]. \end{aligned} \quad (B.4)$$

It is clear from the above expression that the first integral vanishes at infinity (due to Gauss's divergence theorem) and the second integral is equal to zero due to the Euler-Lagrange equations of motion  $\square \bar{C}_1 = 0$  and  $\square \bar{F}^0 = 0$ . Therefore,  $s_b Q_b = -i \{Q_b, Q_b\} = 0$  implies that the BRST charge  $Q_b$  is nilpotent (i.e.  $Q_b^2 = 0$ ) of order two.

In an exactly similar fashion, we prove the nilpotency of the anti-BRST charge  $Q_{ab}$  by exploiting the principle of symmetry transformations where the concept of generator plays an important role. For instance, taking the help from (40) and (34), it can be checked that

$$s_{ab} Q_{ab} = \int d^5x \left[ -(\partial^0 F^i - \partial^i F^0)(\partial_i \bar{C}_2) + (\partial^0 \bar{K}^{\nu\eta} + \partial^\nu \bar{K}^{\eta 0} + \partial^\eta \bar{K}^{0\nu})(\partial_\nu \bar{\beta}_\eta - \partial_\eta \bar{\beta}_\nu) \right]. \quad (B.5)$$

The above expression can be re-expressed by using the following Euler-Lagrange equations of motion [cf. (39)] that emerge from the Lagrangian density  $\mathcal{L}_{(\bar{b})}$ , namely;

$$\frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu \bar{\mathcal{K}}^{\nu\eta}) = \partial_\kappa \bar{K}_{\lambda\rho} + \partial_\lambda \bar{K}_{\rho\kappa} + \partial_\rho \bar{K}_{\kappa\lambda}, \quad \square C_{\mu\nu} + \frac{3}{2}(\partial_\mu F_\nu - \partial_\nu F_\mu) = 0. \quad (B.6)$$

Actually, it can be seen that the above equations lead to

$$\partial^0 \bar{K}^{\nu\eta} + \partial^\nu \bar{K}^{\eta 0} + \partial^\eta \bar{K}^{0\nu} = -\frac{1}{2} \varepsilon^{0\mu\lambda\rho\nu\eta} (\partial_\mu \bar{\mathcal{K}}^{\lambda\rho}), \quad (\partial^0 F^i - \partial^i F^0) = -\frac{2}{3} \square C^{0i}, \quad (B.7)$$

as special cases. Substitution of (B.7) into (B.5), yields

$$s_{ab} Q_{ab} = \int d^5x \partial_i \left[ \frac{2}{3} (\square C^{0i}) \bar{C}_2 - \frac{1}{2} \varepsilon^{0ijklm} \bar{\mathcal{K}}_{jk} (\partial_l \bar{\beta}_m - \partial_m \bar{\beta}_l) \right]$$

$$-\frac{2}{3} \int d^5x \left[ \square (\partial_i C^{0i}) \right] \bar{C}_2. \quad (B.8)$$

This expression can be further simplified by using the equation of motion  $\partial_\mu C^{\mu\nu} - \partial^\nu C_1 + 2F^\nu = 0$  which leads to  $\partial_i C^{0i} = -(\partial^0 C_1) + 2F^0$ . The final form of (B.8) is:

$$\begin{aligned} s_{ab} Q_{ab} = & \int d^5x \partial_i \left[ \frac{2}{3} (\square C^{i0}) \bar{C}_2 - \frac{1}{2} \varepsilon^{0ijklm} \bar{\mathcal{K}}_{jk} (\partial_l \bar{\beta}_m - \partial_m \bar{\beta}_l) \right] \\ & + \frac{2}{3} \int d^5x \left[ \partial^0 (\square C_1) \bar{C}_2 - 2(\square F^0) \bar{C}_2 \right]. \end{aligned} \quad (B.9)$$

The above equation explicitly implies that  $s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} \implies Q_{ab}^2 = 0$  when we use the Euler-Lagrange equations of motion  $\square C_1 = 0$  and  $\square F^0 = 0$  and throw away the total space derivative terms (due to Gauss's divergence theorem).

## Appendix C

Here we provide the key ingredients for the proof of absolute anticommutativity of the conserved (anti-)BRST charges  $Q_{(a)b}$  [cf. (41), (40)] with the help of symmetry transformations [cf. (34), (32)] and the concept of generators for the theory. Let us take as an example the proof of  $s_b Q_{ab} = -i \{Q_{ab}, Q_b\} = 0$ . Using (40) and (32), we obtain

$$\begin{aligned} s_b Q_{ab} = & \int d^5x \left[ \frac{1}{2!} \varepsilon^{0ijklm} (\partial_i K_{jk}) \bar{\mathcal{K}}_{lm} - (\partial^0 K^{\nu\eta} + \partial^\nu K^{\eta 0} + \partial^\eta K^{0\nu}) \bar{K}_{\nu\eta} + \bar{K}^{0i} (\partial_i B_1) \right. \\ & + (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) (\partial_\nu f_\eta - \partial_\eta f_\nu) - (\partial^0 \beta^i - \partial^i \beta^0) (\partial_i B_2) + B_1 \dot{B}_1 \\ & - (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) (\partial_\nu \bar{F}_\eta - \partial_\eta \bar{F}_\nu) + (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i B) + B \dot{B}_2 \\ & \left. + (\partial^0 f^i - \partial^i f^0) \bar{f}_i - (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) F_i + B_2 \dot{B} \right]. \end{aligned} \quad (C.1)$$

The above expression requires some involved algebraic computations in the proof of  $s_b Q_{ab} = -i \{Q_{ab}, Q_b\} = 0$ . Therefore, for the sake of brevity as well as step-by-step computations, we divide the r.h.s. of (C.1) into four parts as given below

$$I_1 = \int d^5x \left[ \frac{1}{2!} \varepsilon^{0ijklm} (\partial_i K_{jk}) \bar{\mathcal{K}}_{lm} - (\partial^0 K^{\nu\eta} + \partial^\nu K^{\eta 0} + \partial^\eta K^{0\nu}) \bar{K}_{\nu\eta} \right], \quad (C.2a)$$

$$I_2 = \int d^5x \left[ \bar{K}^{0i} (\partial_i B_1) + B_1 \dot{B}_1 \right], \quad (C.2b)$$

$$I_3 = \int d^5x \left[ (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i B) - (\partial^0 \beta^i - \partial^i \beta^0) (\partial_i B_2) + B \dot{B}_2 + B_2 \dot{B} \right], \quad (C.2c)$$

$$\begin{aligned} I_4 = & \int d^5x \left[ (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^{\eta 0} + \partial^\eta \bar{C}^{0\nu}) (\partial_\nu f_\eta - \partial_\eta f_\nu) + (\partial^0 f^i - \partial^i f^0) \bar{f}_i \right. \\ & \left. - (\partial^0 C^{\nu\eta} + \partial^\nu C^{\eta 0} + \partial^\eta C^{0\nu}) (\partial_\nu \bar{F}_\eta - \partial_\eta \bar{F}_\nu) - (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) F_i \right]. \end{aligned} \quad (C.2d)$$

Exploiting the CF-type condition  $K_{\mu\nu} + \bar{K}_{\mu\nu} = \partial_\mu \phi_\nu^{(1)} - \partial_\nu \phi_\mu^{(1)}$  [cf. (24)], the second term of the integral  $I_1$  can be re-expressed as

$$- \int d^5x \left( \partial^0 K^{\nu\eta} + \partial^\nu K^{\eta 0} + \partial^\eta K^{0\nu} \right) \bar{K}_{\nu\eta} = \int d^5x \left( \partial^0 \bar{K}^{\nu\eta} + \partial^\nu \bar{K}^{\eta 0} + \partial^\eta \bar{K}^{0\nu} \right) \bar{K}_{\nu\eta}. \quad (C.3)$$

The Euler-Lagrange equations of motion  $\frac{1}{2} \varepsilon_{\mu\nu\eta\kappa\lambda\rho} (\partial^\mu \bar{K}^{\nu\eta}) = \partial_\kappa \bar{K}_{\lambda\rho} + \partial_\lambda \bar{K}_{\rho\kappa} + \partial_\rho \bar{K}_{\kappa\lambda}$  [cf. (39)], derived from the Lagrangian density  $\mathcal{L}_{(\bar{b})}$ , imply

$$\partial^0 \bar{K}^{\nu\eta} + \partial^\nu \bar{K}^{\eta 0} + \partial^\eta \bar{K}^{0\nu} = -\frac{1}{2} \varepsilon^{0\mu\lambda\rho\nu\eta} (\partial_\mu \bar{K}_{\lambda\rho}). \quad (C.4)$$

Substituting (C.4) into (C.2a) and performing partial integration, we obtain

$$I_1 = -\frac{1}{2!} \int d^5x \partial_i \left( \varepsilon^{0ijklm} \bar{K}_{jk} \bar{K}_{lm} \right) + \frac{1}{2!} \int d^5x \varepsilon^{0ijklm} \left[ \partial_i (K_{jk} + \bar{K}_{jk}) \right] \bar{K}_{lm}. \quad (C.5)$$

Thus, we note that  $I_1 = 0$  if we use (24) and if we assume that the physical fields are those that vanish at infinity (due to Gauss's divergence theorem).

Let us now take the integral  $I_2$ . Performing the partial integration, we obtain

$$I_2 = \int d^5x \partial_i \left( B_1 \bar{K}^{0i} \right) - \int d^5x \left[ B_1 (\partial_i \bar{K}^{0i}) - B_1 \dot{B}_1 \right]. \quad (C.6)$$

It should be noted that the first term goes to zero (due to the validity of Gauss's divergence theorem) and the second term vanishes if we use the equation of motion  $\partial_\mu \bar{K}^{\mu\nu} + \partial^\nu B_1 = 0$  from (39) (which implies  $\partial_i \bar{K}^{0i} = \dot{B}_1$ ). If we carry out the partial integration and throw away the total space derivative terms, we obtain the form of integral  $I_3$  as

$$I_3 = - \int d^5x \left[ (\partial^0 \partial_i \bar{\beta}^i - \partial_i \partial^i \bar{\beta}^0) B - (\partial^0 \partial_1 \beta^i - \partial_i \partial^i \beta^0) B_2 - B \dot{B}_2 - B_2 \dot{B} \right]. \quad (C.7)$$

Using the appropriate equations of motion  $B = -(\partial \cdot \beta)$ ,  $B_2 = (\partial \cdot \bar{\beta})$  from (39) [which imply  $\partial_i \beta^i = -B - \partial_0 \beta^0$ ,  $\partial_i \bar{\beta}^i = -\partial_0 \bar{\beta}^0 + B_2$ ], the above expression can be re-expressed as

$$I_3 = \int d^5x \left[ B (\square \bar{\beta}^0) - B_2 (\square \beta^0) \right]. \quad (C.8)$$

It is clear from the above equation that the integral  $I_3$  vanishes ( $I_3 = 0$ ) if we use the Euler-Lagrange equations of motion  $\square \beta_0 = 0$ ,  $\square \bar{\beta}_0 = 0$  [cf. (39)].

The integral  $I_4$  can be expressed in the component form as

$$\begin{aligned} I_4 = & -2 \int d^5x \left[ \partial^0 (\partial_i \bar{C}^{ij}) f_j + \partial_i (\partial^i \bar{C}^{j0} - \partial^j \bar{C}^{i0}) f_j - \frac{1}{2} (\partial^0 f^i - \partial^i f^0) \bar{f}_i \right. \\ & \left. - \partial^0 (\partial_i C^{ij}) \bar{F}_j - \partial_i (\partial^i C^{j0} - \partial^j C^{i0}) \bar{F}_j + \frac{1}{2} (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) F_i \right], \end{aligned} \quad (C.9)$$

where we have performed the partial integration in the first and third terms of the integral  $I_4$  and have thrown away the total space derivative terms. Using the following equations

of motion  $\partial_\mu C^{\mu\nu} - \partial^\nu C_1 + 2F^\nu = 0$ ,  $\partial_\mu \bar{C}^{\mu\nu} - \partial^\nu \bar{C}_1 + 2\bar{f}^\nu = 0$ , the above integral can be reduced to the following form:

$$I_4 = 2 \int d^5x \left[ (\square \bar{C}^{0i}) f_i - (\square C^{0i}) \bar{F}_i + 2(\partial^0 \bar{f}^i - \partial^i \bar{f}^0) f_i - 2(\partial^0 F^i - \partial^i F^0) \bar{F}_i \right. \\ \left. + \frac{1}{2} (\partial^0 f^i - \partial^i f^0) \bar{f}_i - \frac{1}{2} (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) F_i \right]. \quad (C.10)$$

Further, exploiting the appropriate equations of motion  $\square C^{0i} = -\frac{3}{2}(\partial^0 F^i - \partial^i F^0)$ ,  $\square \bar{C}^{0i} = -\frac{3}{2}(\partial^0 \bar{f}^i - \partial^i \bar{f}^0)$  [cf. (39)] and the CF-type conditions  $f_\mu + F_\mu = \partial_\mu C_1$ ,  $\bar{f}_\mu + \bar{F}_\mu = \partial_\mu \bar{C}_1$  from (43), the above expression can be re-written as

$$I_4 = \int d^5x \left[ (\partial^0 \bar{f}^i - \partial^i \bar{f}^0) (f_i + F_i) - (\partial^0 F^i - \partial^i F^0) (\bar{f}_i + \bar{F}_i) \right] \\ \equiv -\frac{2}{3} \int d^5x \left[ (\square \bar{C}^{0i}) (\partial_i C_1) - (\square C^{0i}) (\partial_i \bar{C}_1) \right]. \quad (C.11)$$

After performing the partial integration and, then, using the Euler-Lagrange equations of motion  $\partial_i C^{0i} = -\partial^0 C_1 + 2F^0$ ,  $\partial_i \bar{C}^{0i} = -\partial^0 \bar{C}_1 + 2\bar{f}^0$  [cf. (39)], the above integral can be proved to be zero. The following form of  $I_4$  corroborates the above statement, namely;

$$I_4 = -\frac{2}{3} \int d^5x \left[ \partial^0 (\square \bar{C}_1) C_1 - 2(\square \bar{f}^0) C_1 - \partial^0 (\square C_1) \bar{C}_1 + 2(\square F^0) \bar{C}_1 \right] = 0, \quad (C.12)$$

where we have thrown away the total space derivative terms and we have used the equations of motion  $\square C_1 = 0$ ,  $\square \bar{C}_1 = 0$ ,  $\square F^0 = 0$ ,  $\square \bar{f}^0 = 0$  [cf. (39)].

From the above computations, it is clear that  $s_b Q_{ab} = -i \{Q_{ab}, Q_b\} = 0$  shows that  $Q_b$  and  $Q_{ab}$  are anticommuting in nature on the constrained hypersurface defined by the CF-type equations (24) and (43). In an exactly similar fashion, one can also prove that  $s_{ab} Q_b = -i \{Q_b, Q_{ab}\} = 0$  (which imply the anticommutativity of  $Q_b$  and  $Q_{ab}$ ). We would like to comment that the complete extended BRST algebra (79) can be derived by using the concept of principle of symmetry transformations as we have demonstrated in the proof of nilpotency ( $Q_{(a)b}^2 = 0$ ) and anticommutativity property ( $\{Q_b, Q_{ab}\} = 0$ ) (cf. Appendices B and C). In this method of proof, the algebra is straightforward and handy.

## Appendix D

Here we discuss the discrete symmetry transformations for the Abelian  $p$ -form gauge theory in  $D = 2p$  dimensions of spacetime. It is clear that the kinetic term of this gauge theory will be constructed from the  $(p+1)$ -form  $F^{(p+1)}$  as given below

$$F^{(p+1)} = d A^{(p)} = \left[ \frac{dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \dots \wedge dx^{\mu_{p+1}}}{(p+1)!} \right] F_{\mu_1 \mu_2 \mu_3 \dots \mu_{p+1}}, \quad (D.1)$$

where  $F_{\mu_1 \mu_2 \mu_3 \dots \mu_{p+1}}$  is the totally antisymmetric curvature tensor that is expressed in terms of the antisymmetrized version of derivatives on the  $p$ -form potential  $A_{\mu_1 \mu_2 \mu_3 \dots \mu_p}$ . The

gauge-fixing term in the  $D = 2p$  dimensional spacetime for the  $p$ -form gauge potential is a  $(p-1)$ -form antisymmetric tensor (defined in terms of the co-exterior derivative  $\delta$ ) as

$$\delta A^{(p)} = - * d * A^{(p)} = \left[ \frac{dx^{\mu_2} \wedge dx^{\mu_3} \dots \wedge dx^{\mu_p}}{(p-1)!} \right] \left( \partial^{\mu_1} A_{\mu_1 \mu_2 \mu_3 \dots \mu_p} \right), \quad (D.2)$$

where  $(\partial^{\mu_1} A_{\mu_1 \mu_2 \mu_3 \dots \mu_p})$  is totally antisymmetric in all the indices from  $\mu_2$  to  $\mu_p$ .

The gauge-fixed Lagrangian density for the  $D = 2p$  dimensional Abelian  $p$ -form gauge theory can be written (in the Feynman gauge) as (see, e.g., [36])

$$\begin{aligned} \mathcal{L} = & \frac{1}{2(p+1)!} \left( F_{\mu_1 \mu_2 \mu_3 \dots \mu_{p+1}} \right) \left( F^{\mu_1 \mu_2 \mu_3 \dots \mu_{p+1}} \right) \\ & + \frac{1}{2(p-1)!} \left( \partial^{\mu_1} A_{\mu_1 \mu_2 \mu_3 \dots \mu_p} \right) \left( \partial_{\nu_1} A^{\nu_1 \mu_2 \mu_3 \dots \mu_p} \right). \end{aligned} \quad (D.3)$$

The above Lagrangian density would respect (dual-)gauge symmetry transformations analogous to (2), (7) and (11). The key reason behind the existence of the dual-gauge symmetry transformations is the following *self-duality* condition

$$* A^{(p)} = \frac{(-1)^p}{p!} \varepsilon_{\mu_1 \mu_2 \mu_3 \dots \mu_p \mu_{p+1} \dots \mu_{2p}} A^{\mu_{p+1} \mu_{p+2} \dots \mu_{2p}}, \quad (D.4)$$

where the Hodge duality  $(*)$  operation is defined on a  $2p$ -dimensional flat spacetime manifold, on which, a  $2p$ -dimensional Levi-Civita tensor  $\varepsilon_{\mu_1 \mu_2 \mu_3 \dots \mu_p \mu_{p+1} \dots \mu_{2p}}$  can exist. The following discrete symmetry transformations on the  $p$ -form gauge potential, namely;

$$A_{\mu_1 \mu_2 \dots \mu_p} \longrightarrow \pm \frac{i}{p!} \varepsilon_{\mu_1 \mu_2 \dots \mu_p \mu_{p+1} \dots \mu_{2p}} A^{\mu_{p+1} \mu_{p+2} \dots \mu_{2p}}, \quad (D.5)$$

would turn out to be the *symmetry* transformations for the gauge-fixed Lagrangian density (D.3) as we have seen the existence of analogous symmetries in the cases of 2D Abelian 1-form, 4D Abelian 2-form and 6D Abelian 3-form gauge theories. In fact, under the discrete transformations (D.5), the kinetic and gauge-fixing terms would exchange with each-other in the gauge-fixed Lagrangian density (D.3) in the Feynman gauge.

The “classical” dual-gauge symmetries can be generalized to the nilpotent “quantum” (anti-)dual-BRST symmetries in the same way as we have done for the 2D Abelian 1-form, 4D Abelian 2-form and 6D Abelian 3-form gauge theories. We very briefly outline here the derivation of, first of all, the proper (anti-)BRST symmetry transformations by exploiting the geometrical superfield approach [28,29]. Primarily, we invoke here the following horizontality condition  $(\tilde{F}^{(p+1)} = F^{(p+1)})$ . In other words, we demand

$$\tilde{d} \tilde{A}^{(p)} = d A^{(p)}, \quad (D.6)$$

where  $d = dx^\mu \partial_\mu$  ( $\mu = 0, 1, 2, \dots, 2p-1$ ) is the ordinary exterior derivative defined on the ordinary  $D = 2p$  dimensional spacetime and  $A^{(p)}$  is the ordinary  $p$ -form connection. On the l.h.s. of (D.6), we have the super exterior derivative  $\tilde{d}$  (with  $\tilde{d}^2 = 0$ ) and super



$p$ -form connection  $\tilde{A}^{(p)}$  which are the generalizations of their ordinary counterparts (in  $2p$ -dimensions of spacetime) onto a  $(2p, 2)$ -dimensional supermanifold. These generalizations can be succinctly expressed, in the mathematical form, as

$$d \longrightarrow \tilde{d} = dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}},$$

$$A^{(p)} \longrightarrow \tilde{A}^{(p)} = \left( \frac{dZ^{M_1} \wedge dZ^{M_2} \dots \wedge dZ^{M_p}}{p!} \right) \tilde{A}_{M_1 M_2 \dots M_p}(x, \theta, \bar{\theta}), \quad (D.7)$$

where  $Z^M = (x^\mu, \theta, \bar{\theta})$  and  $\partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$  are the superspace variables and corresponding derivatives that characterize the  $(2p, 2)$ -dimensional supermanifold on which the Abelian  $p$ -form gauge theory has been generalized. Here the bosonic variable  $x^\mu$  ( $\mu = 0, 1, 2, \dots, 2p-1$ ) are the ordinary spacetime variable and  $(\theta, \bar{\theta})$  are a pair of Grassmannian variables (with  $\theta^2 = \bar{\theta}^2 = 0$ ,  $\theta\bar{\theta} + \bar{\theta}\theta = 0$ ). Furthermore, the multiplet superfields  $\tilde{A}_{M_1 M_2 \dots M_{p-1} M_p}(x, \theta, \bar{\theta})$  will have (anti)symmetric components (i.e.  $\tilde{A}_{\mu_1 \mu_2 \dots \mu_p}$ ,  $\tilde{A}_{\mu_1 \mu_2, \theta, \bar{\theta} \dots \mu_p}$ , etc.). These superfields  $\tilde{A}_{M_1 M_2 \dots M_p}(x, \theta, \bar{\theta})$  can be expanded along the Grassmannian directions of the  $(2p, 2)$ -dimensional supermanifold (see, e.g., [21]) in terms of the basic, auxiliary and secondary fields of the ordinary  $2p$ -dimensional (anti-)BRST invariant field theory.

At this juncture, the horizontality condition (D.6) plays a decisive role. This condition is basically the covariant reduction of the supercurvature  $(p+1)$ -form  $\tilde{F}^{(p+1)}$  to the ordinary curvature  $(p+1)$ -form  $F^{(p+1)}$ . In fact, first of all, we substitute the values of  $\tilde{d}$  and  $\tilde{A}^{(p)}$  from (D.7) into the l.h.s. of (D.6). The covariant reduction of the supercurvature  $(p+1)$ -form  $\tilde{F}^{(p+1)}$  (defined on the  $(2p, 2)$ -dimensional supermanifold) to the ordinary curvature  $(p+1)$ -form  $F^{(p+1)}$  (defined on  $2p$ -dimensional supermanifold), leads to the derivation of off-shell nilpotent ( $s_{(a)b}^2 = 0$ ) and absolutely anticommuting ( $s_b s_{ab} + s_{ab} s_b = 0$ ) (anti-)BRST transformations (see, e.g., [21] for Abelian 2-form and 3-form gauge theories). From these proper (anti-)BRST symmetry transformations, one can always obtain the (anti-)BRST invariant coupled Lagrangian densities (see, e.g., Appendix A for 3-form gauge theory) that incorporate the gauge-fixing and Faddeev-Popov ghost terms. Furthermore, these Lagrangian densities would always respect the (anti-)BRST symmetry transformations derived from the horizontality condition (within the framework of superfield formalism [28,29]).

Finally, the discrete symmetry transformations for the gauge field (D.5) can be generalized to incorporate such discrete symmetry transformations on the (anti-)ghost and auxiliary fields of the theory. The full discrete symmetry transformations define the analogue of the Hodge duality  $(*)$  operation of differential geometry. As a consequence, we can define the analogue of relation (77) that would enable us to deduce the (anti-)dual-BRST symmetry transformations  $s_{(a)d}$ . The latter symmetries also turn out to be off-shell nilpotent ( $s_{(a)d}^2 = 0$ ) and absolutely anticommuting ( $s_d s_{ad} + s_{ad} s_d = 0$ ) in nature. Having obtained these basic nilpotent ( $s_{(a)b}^2 = 0$ ,  $s_{(a)d}^2 = 0$ ) (anti-)BRST and (anti-)co-BRST symmetry transformations  $s_{(a)b}$  and  $s_{(a)d}$ , we can derive bosonic as well as ghost-scale symmetries of the theory and show that the Abelian  $p$ -form gauge theory, in  $D = 2p$  dimensions of spacetime, provides a model for the Hodge theory where we obtain the physical realizations of all the cohomological quantities of differential geometry [17-20].

## Appendix E

Here we show that the Lagrangian densities (30) and (31), in addition to the six continuous symmetries  $(s_{(a)b}, s_{(a)d}, s_\omega, s_g)$ , respect two *new bosonic* symmetries but these new bosonic symmetries [cf. (67)] can not be regarded as the analogue of Laplacian operator of differential geometry. These new bosonic symmetries are defined as:  $s_\omega^{(1)} = \{s_b, s_{ad}\}$ ,  $s_\omega^{(2)} = \{s_d, s_{ab}\}$ . It is interesting to point out that these bosonic symmetries  $s_\omega^{(1)}$  and  $s_\omega^{(2)}$  do not exist (i.e.  $\{s_d, s_{ab}\} = \{s_b, s_{ad}\} = 0$ ) for the 2D Abelian 1-form [10-12] and 4D Abelian 2-form gauge theories [13-16]. Under the following bosonic transformations  $s_\omega^{(1)}$  and  $s_\omega^{(2)}$ :

$$s_\omega^{(1)}\phi_\mu^{(1)} = -\partial_\mu B, \quad s_\omega^{(1)}\phi_\mu^{(2)} = -\partial_\mu B, \quad s_\omega^{(1)}\bar{\beta}_\mu = \partial_\mu(B_1 - B_3), \quad s_\omega^{(1)}[A_{\mu\nu\eta}, K_{\mu\nu}, \bar{K}_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}, C_{\mu\nu}, \bar{C}_{\mu\nu}, F_\mu, \bar{F}_\mu, f_\mu, \bar{f}_\mu, \beta_\mu, B, B_1, B_2, B_3, C_1, \bar{C}_1, C_2, \bar{C}_2] = 0, \quad (E.1)$$

$$s_\omega^{(2)}\phi_\mu^{(1)} = \partial_\mu B_2, \quad s_\omega^{(2)}\phi_\mu^{(2)} = -\partial_\mu B_2, \quad s_\omega^{(2)}\beta_\mu = \partial_\mu(B_1 + B_3), \quad s_\omega^{(2)}[A_{\mu\nu\eta}, K_{\mu\nu}, \bar{K}_{\mu\nu}, \mathcal{K}_{\mu\nu}, \bar{\mathcal{K}}_{\mu\nu}, C_{\mu\nu}, \bar{C}_{\mu\nu}, F_\mu, \bar{F}_\mu, f_\mu, \bar{f}_\mu, \beta_\mu, B, B_1, B_2, B_3, C_1, \bar{C}_1, C_2, \bar{C}_2] = 0, \quad (E.2)$$

the Lagrangian densities (30) and (31) transform as follows:

$$s_\omega^{(1)}\mathcal{L}_{(b)} = s_\omega^{(1)}\mathcal{L}_{(\bar{b})} = -\partial_\mu[(B_1 - B_3)(\partial^\mu B) - B \partial^\mu(B_1 - B_3)], \quad (E.3)$$

$$s_\omega^{(2)}\mathcal{L}_{(b)} = s_\omega^{(2)}\mathcal{L}_{(\bar{b})} = \partial_\mu[(B_1 + B_3)(\partial^\mu B_2) - B_2 \partial^\mu(B_1 + B_3)]. \quad (E.4)$$

Furthermore, it can be checked that the above bosonic symmetry transformations  $s_\omega^{(1)}$  and  $s_\omega^{(2)}$  lead to the following conserved currents

$$J_{(\omega, b)}^\mu{}^{(1)} = (\mathcal{K}^{\mu\nu} - K^{\mu\nu}) (\partial_\nu B) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \partial_\nu(B_1 - B_3), \quad (E.5)$$

$$J_{(\omega, \bar{b})}^\mu{}^{(1)} = (\bar{\mathcal{K}}^{\mu\nu} - \bar{K}^{\mu\nu}) (\partial_\nu B) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \partial_\nu(B_1 - B_3), \quad (E.6)$$

$$J_{(\omega, b)}^\mu{}^{(2)} = (\mathcal{K}^{\mu\nu} + K^{\mu\nu}) (\partial_\nu B_2) - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \partial_\nu(B_1 + B_3), \quad (E.7)$$

$$J_{(\omega, \bar{b})}^\mu{}^{(2)} = (\bar{\mathcal{K}}^{\mu\nu} + \bar{K}^{\mu\nu}) (\partial_\nu B_2) - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \partial_\nu(B_1 + B_3), \quad (E.8)$$

where the conservation law can be proven by exploiting the Euler-Lagrange equations of motion (38) and (39). The above conserved currents lead to the following expressions for the conserved charges  $(Q_{(\omega, s)}^{(r)} = \int d^5x J_{(\omega, s)}^{0(r)})$  where  $r = 1, 2$  and  $s = b, \bar{b}$

$$Q_{(\omega, b)}^{(1)} = \int d^5x [(\mathcal{K}^{0i} - K^{0i}) (\partial_i B) - (\partial^0 \beta^i - \partial^i \beta^0) \partial_i(B_1 - B_3)], \quad (E.9)$$

$$Q_{(\omega, \bar{b})}^{(1)} = \int d^5x [(\bar{\mathcal{K}}^{0i} - \bar{K}^{0i}) (\partial_i B) - (\partial^0 \beta^i - \partial^i \beta^0) \partial_i(B_1 - B_3)], \quad (E.10)$$

$$Q_{(\omega, b)}^{(2)} = \int d^5x [(\mathcal{K}^{0i} + K^{0i}) (\partial_i B_2) - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i(B_1 + B_3)], \quad (E.11)$$

$$Q_{(\omega, \bar{b})}^{(2)} = \int d^5x [(\bar{\mathcal{K}}^{0i} + \bar{K}^{0i}) (\partial_i B_2) - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i(B_1 + B_3)]. \quad (E.12)$$

Even though, the above bosonic charges remain invariant under the (anti-)BRST as well as (anti-)dual BRST symmetry transformations [cf. (34), (32), (49) and (48)], under the ghost-scale symmetry transformations  $s_g$  [cf. (72)], we obtain the following

$$s_g Q_{(\omega,b)}^{(1)} = -i[Q_{(\omega,b)}^{(1)}, Q_g] = +2 Q_{(\omega,b)}^{(1)}, \quad s_g Q_{(\omega,\bar{b})}^{(1)} = -i[Q_{(\omega,\bar{b})}^{(1)}, Q_g] = +2 Q_{(\omega,\bar{b})}^{(1)}, \quad (E.13)$$

$$s_g Q_{(\omega,b)}^{(2)} = -i[Q_{(\omega,b)}^{(2)}, Q_g] = -2 Q_{(\omega,b)}^{(2)}, \quad s_g Q_{(\omega,\bar{b})}^{(2)} = -i[Q_{(\omega,\bar{b})}^{(2)}, Q_g] = -2 Q_{(\omega,\bar{b})}^{(2)}. \quad (E.14)$$

Thus, the ghost numbers of  $Q_{(\omega,b)}^{(1)}$ ,  $Q_{(\omega,\bar{b})}^{(1)}$  are (+2) and the ghost numbers of  $Q_{(\omega,b)}^{(2)}$ ,  $Q_{(\omega,\bar{b})}^{(2)}$  are (-2) [cf. (E.17) and (E. 18) below] and, as a result, they *do not* commute with the ghost charge of the present theory. Moreover, at the level of symmetry transformations, the bosonic symmetries  $s_\omega^{(1)}$  and  $s_\omega^{(2)}$  commute with  $s_b, s_{ab}, s_d, s_{ad}$  but they *do not* commute with the ghost-scale transformation  $s_g$  for the generic fields  $\Psi_1$  and  $\Psi_2$ , namely;

$$[s_\omega^{(1)}, s_g] \Psi_1 \neq 0, \quad \Psi_1 = \phi_\mu^{(1)}, \phi_\mu^{(2)}, \bar{\beta}_\mu, \quad (E.15)$$

$$[s_\omega^{(2)}, s_g] \Psi_2 \neq 0, \quad \Psi_2 = \phi_\mu^{(1)}, \phi_\mu^{(2)}, \beta_\mu, \quad (E.16)$$

However, the bosonic symmetry  $s_\omega$  [cf. (58)] commutes with all the symmetry transformations  $s_b, s_{ab}, s_d, s_{ad}, s_g$ . Thus, due to the above mentioned reasons, these bosonic symmetries  $s_\omega^{(1)}$  and  $s_\omega^{(2)}$  (and the corresponding generators  $Q_\omega^{(1)}$  and  $Q_\omega^{(2)}$ ) are *not* the Casimir operators for the algebras (75) and (79). It should be noted that, for the sake of brevity, we have taken  $Q_\omega^{(1)} \equiv (Q_{(\omega,b)}^{(1)}, Q_{(\omega,\bar{b})}^{(1)})$  and  $Q_\omega^{(2)} \equiv (Q_{(\omega,b)}^{(2)}, Q_{(\omega,\bar{b})}^{(2)})$  for our further discussions.

The clinching evidence that  $Q_\omega^{(1)}$  and  $Q_\omega^{(2)}$  are *not* the physical realization of the Laplacian operator of differential geometry emerges out from the ghost number considerations. From the algebra in (E.13) and (E.14), it is obvious that

$$i Q_g Q_\omega^{(1)} |\chi\rangle_n = (n+2) Q_\omega^{(1)} |\chi\rangle_n, \quad (E.17)$$

$$i Q_g Q_\omega^{(2)} |\chi\rangle_n = (n-2) Q_\omega^{(2)} |\chi\rangle_n. \quad (E.18)$$

Thus, we establish that  $Q_\omega^{(1)}$  and  $Q_\omega^{(2)}$  are *not* the analogue of Laplacian operator as is evident from the comparison between [(E.17), (E.18)] and (78).

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